On the Γ-limit of singular perturbation problems with optimal profiles which are not one-dimensional. Part I: The upper bound

November 27, 2012

Arkady Poliakovsky [1]

Università di Roma 'Tor Vergata', Dipartimento di Matematica,
Via della Ricerca Scientifica, 1, 00133 Rome, Italy

Abstract

In Part I we construct an upper bound, in the spirit of Γ- lim sup, achieved by multidimensional profiles, for some general classes of singular perturbation problems, with or without the prescribed differential constraint, taking the form

$$E_\varepsilon(v) := \int_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon^n \nabla^n v, ..., \varepsilon \nabla v, v\Big) dx \quad \text{for} v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^k \text{ such that } A \cdot \nabla v = 0,$$

where the function $F \geq 0$ and $A : \mathbb{R}^{k\times N} \to \mathbb{R}^m$ is a prescribed linear operator (for example, $A :\equiv 0$, $A \cdot \nabla v :=$ curl v and $A \cdot \nabla v = \operatorname{div} v$) which includes, in particular, the problems considered in [27]. This bound is in general sharper then one obtained in [27].

1 Introduction

Definition 1.1. Consider a family $\{I_\varepsilon\}_{\varepsilon>0}$ of functionals $I_\varepsilon(\phi) : U \to [0, +\infty]$, where U is a given metric space. The Γ-limits of I_ε are defined by:

$$(\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon)(\phi) := \inf\left\{\liminf_{\varepsilon\to 0^+} I_\varepsilon(\phi_\varepsilon) : \ \{\phi_\varepsilon\}_{\varepsilon>0} \subset U,\ \phi_\varepsilon \to \phi \text{ in } U \text{ as } \varepsilon \to 0^+\right\},$$

$$(\Gamma - \limsup_{\varepsilon\to 0^+} I_\varepsilon)(\phi) := \inf\left\{\limsup_{\varepsilon\to 0^+} I_\varepsilon(\phi_\varepsilon) : \ \{\phi_\varepsilon\}_{\varepsilon>0} \subset U,\ \phi_\varepsilon \to \phi \text{ in } U \text{ as } \varepsilon \to 0^+\right\},$$

$$(\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon)(\phi) := (\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon\big)(\phi) = (\Gamma - \limsup_{\varepsilon\to 0^+} I_\varepsilon)(\phi) \quad \text{in the case they are equal.}$$

It is useful to know the Γ-limit of I_ε, because it describes the asymptotic behavior as $\varepsilon \downarrow 0$ of minimizers of I_ε, as it is clear from the following simple statement:

Proposition 1.1 (De-Giorgi)**.** *Assume that ϕ_ε is a minimizer of I_ε for every $\varepsilon > 0$. Then:*

- *If $I_0(\phi) = (\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon)(\phi)$ and $\phi_\varepsilon \to \phi_0$ as $\varepsilon \to 0^+$ then ϕ_0 is a minimizer of I_0.*
- *If $I_0(\phi) = (\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon)(\phi)$ (i.e. it is a full Γ-limit of $I_\varepsilon(\phi)$) and for some subsequence $\varepsilon_n \to 0^+$ as $n \to \infty$, we have $\phi_{\varepsilon_n} \to \phi_0$, then ϕ_0 is a minimizer of I_0.*

Usually, for finding the Γ-limit of $I_\varepsilon(\phi)$, we need to find two bounds.

(*) Firstly, we find a lower bound, i.e. a functional $\underline{I}(\phi)$ such that for every family $\{\phi_\varepsilon\}_{\varepsilon>0}$, satisfying $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0^+$, we have $\liminf_{\varepsilon\to 0^+} I_\varepsilon(\phi_\varepsilon) \geq \underline{I}(\phi)$.

[1] E-mail: poliakov@mat.uniroma2.it

(**) Secondly, we find an upper bound, i.e. a functional $\overline{I}(\phi)$, such that for every $\phi \in U$ there exists a family $\{\psi_\varepsilon\}_{\varepsilon>0}$, satisfying $\psi_\varepsilon \to \phi$ as $\varepsilon \to 0^+$ and $\limsup_{\varepsilon\to 0^+} I_\varepsilon(\psi_\varepsilon) \le \overline{I}(\phi)$.

(***) If we find that $\underline{I}(\phi) = \overline{I}(\phi) := I(\phi)$, then $I(\phi)$ is the Γ-limit of $I_\varepsilon(\phi)$.

In various applications we deal with the asymptotic behavior as $\varepsilon \to 0^+$ of a family of functionals $\{I_\varepsilon\}_{\varepsilon>0}$ of the following forms.

- In the case of the first order problem the functional I_ε, which acts on functions $\psi : \Omega \to \mathbb{R}^m$, has the form

$$I_\varepsilon(\psi) = \int_\Omega \varepsilon\big|\nabla\psi(x)\big|^2 + \frac{1}{\varepsilon}W\Big(\psi(x), x\Big)\,dx\,, \tag{1.1}$$

  or more generally

$$I_\varepsilon(\psi) = \int_\Omega \frac{1}{\varepsilon}G\Big(\varepsilon^n\nabla^n\psi, \dots, \varepsilon\nabla\psi, \psi, x\Big)\,dx + \int_\Omega \frac{1}{\varepsilon}W\big(\psi, x\big)\,dx\,, \tag{1.2}$$

  where $G(0,\dots,0,\psi,x) \equiv 0$.

- In the case of the second order problem the functional I_ε, which acts on functions $v : \Omega \to \mathbb{R}^k$, has the form

$$I_\varepsilon(v) = \int_\Omega \varepsilon\big|\nabla^2 v(x)\big|^2 + \frac{1}{\varepsilon}W\Big(\nabla v(x), v(x), x\Big)\,dx\,, \tag{1.3}$$

  or more generally

$$I_\varepsilon(v) = \int_\Omega \frac{1}{\varepsilon}G\Big(\varepsilon^n\nabla^{n+1}v, \dots, \varepsilon\nabla^2 v, \nabla v, v, x\Big)\,dx + \int_\Omega \frac{1}{\varepsilon}W\big(\nabla v, v, x\big)\,dx\,, \tag{1.4}$$

  where $G(0,\dots,0,\nabla v, v, x) \equiv 0$.

In this paper we deal with the asymptotic behavior as $\varepsilon \to 0^+$ of a family of functionals of the following general form: Let $\Omega \subset \mathbb{R}^N$ be an open set. For every $\varepsilon > 0$ consider the general functional

$$I_\varepsilon(v) = \big\{I_\varepsilon(\Omega)\big\}(v) := \int_\Omega \frac{1}{\varepsilon}G\Big(\varepsilon^n\nabla^n v, \dots, \varepsilon\nabla v, v, x\Big) + \frac{1}{\varepsilon}W\big(v, x\big)\,dx \quad \text{with } v := (\nabla u, h, \psi),$$
$$\text{where } u \in W^{(n+1),1}_{loc}(\Omega, \mathbb{R}^k),\;\; h \in W^{n,1}_{loc}(\Omega, \mathbb{R}^{d\times N}) \text{ s.t. } \operatorname{div} h \equiv 0,\;\; \psi \in W^{n,1}_{loc}(\Omega, \mathbb{R}^m). \tag{1.5}$$

Here

$$G : \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N^n} \times \ldots \times \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N} \times \mathbb{R}^{\{k\times N\}+\{d\times N\}+m} \times \mathbb{R}^N \to \mathbb{R}$$

and $W : \mathbb{R}^{\{k\times N\}+\{d\times N\}+m} \times \mathbb{R}^N \to \mathbb{R}$ are nonnegative continuous functions and G satisfies $G(0,\dots,0,v,x) \equiv 0$. The functionals in (1.1),(1.2) and (1.3),(1.4) are important particular cases of the general energy I_ε in (1.5). In the general form (1.5) we also include the dependence on div-free function h, which can be useful in the study of problems with non-local terms as the Rivière-Serfaty functional and other functionals in Micromagnetics.

The functionals of the form (1.1) arise in the theories of phase transitions and minimal surfaces. They were first studied by Modica and Mortola [25], Modica [24], Sternberg [38] and others. The Γ-limit of the functional in (1.1), where W does not depend on x explicitly, was obtained in the general vectorial case by Ambrosio in [2]. The Γ-limit of the functional of the form (1.2), where $n = 1$ and there exist $\alpha, \beta \in \mathbb{R}^m$ such that $W(h, x) = 0$ if and only if $h \in \{\alpha, \beta\}$, under some restriction on the explicit dependence on x of G and W, was obtained by Fonseca and Popovici in [16]. The Γ-limit of the functional of the form (1.2), with $n = 2$, $G(\cdot)/\varepsilon \equiv \varepsilon^3|\nabla^2\psi|^2$ and W which doesn't depend on x explicitly, was found by I. Fonseca and C. Mantegazza in [15].

The functionals of second order of the form (1.3) arise, for example, in the gradient theory of solid-solid phase transitions, where one considers energies of the form

$$I_\varepsilon(v) = \int_\Omega \varepsilon|\nabla^2 v(x)|^2 + \frac{1}{\varepsilon}W\Big(\nabla v(x)\Big)\,dx\,, \tag{1.6}$$

where $v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ stands for the deformation, and the free energy density $W(F)$ is nonnegative and satisfies

$$W(F) = 0 \quad \text{if and only if} \quad F \in K := SO(N)A \cup SO(N)B\,.$$

Here A and B are two fixed, invertible matrices, such that $rank(A - B) = 1$ and $SO(N)$ is the set of rotations in $\mathbb{R}^N$. The simpler case where $W(F) = 0$ if and only if $F \in \{A, B\}$ was studied by Conti, Fonseca and Leoni in [10]. The case of problem (1.6), where $N = 2$ and $W(QF) = W(F)$ for all $Q \in SO(2)$ was investigated by Conti and Schweizer in [9] (see also [8] for a related problem). Another important example of the second order energy is the so called Aviles-Giga functional, defined on scalar valued functions v by

$$\int_\Omega \varepsilon|\nabla^2 v|^2 + \frac{1}{\varepsilon}\big(1 - |\nabla v|^2\big)^2 \qquad \text{(see [3],[5],[6]).} \tag{1.7}$$

The main contribution of this work is to improve our method (see [31],[27]) for finding upper bounds in the sense of (**) for the general functional (1.5) in the case where the limiting function belongs to BV-space. In order to formulate the main results of this paper we present the following definitions.

First of all, in order to simplify the notations in (1.5), for every open $\mathcal{U} \subset \mathbb{R}^N$ consider

$$\mathcal{B}(\mathcal{U}) := \Big\{ v \in L^1_{loc}\big(\mathcal{U}, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) : \;\; v = (\nabla u, h, \psi),\;\; u \in W^{1,1}_{loc}(\mathcal{U}, \mathbb{R}^k),$$
$$h \in L^1_{loc}(\mathcal{U}, \mathbb{R}^{d\times N}) \text{ s.t. div } h \equiv 0 \text{ in the sense of distributions},\;\; \psi \in L^1_{loc}(\mathcal{U}, \mathbb{R}^m)\Big\}, \tag{1.8}$$

and

$$F\Big(\nabla^n v, \ldots, \nabla v, v, x\Big) \;:=\; G\Big(\nabla^n v, \ldots, \nabla v, v, x\Big) \;+\; W(v, x) \tag{1.9}$$

Then

$$I_\varepsilon(v) = \int_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon^n \nabla^n v, \ldots, \varepsilon \nabla v, v, x\Big)dx \quad \text{with } v \in \mathcal{B}(\Omega) \cap W^{n,1}_{loc}\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big). \tag{1.10}$$

What can we expect as the Γ-limit or at least as an upper bound of these general energies in the L^p-topology. It is clear that if G and W are nonnegative and W is a continuous on the argument v function, then the upper bound for $I_\varepsilon(\cdot)$ will be finite only if

$$W\big(v(x), x\big) = 0 \quad \text{for a.e. } \; x \in \Omega\,, \tag{1.11}$$

i.e. if we define

$$\mathcal{A}_0 := \Big\{ v \in L^p\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) \cap \mathcal{B}(\Omega) : \;\; W\big(v(x), x\big) = 0 \; \text{ for a.e. } \; x \in \Omega \Big\} \tag{1.12}$$

and

$$\mathcal{A} := \Big\{ v \in L^p\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) : \;\; (\Gamma - \limsup_{\varepsilon\to 0^+} I_\varepsilon)(v) < +\infty \Big\}, \tag{1.13}$$

then clearly $\mathcal{A} \subset \mathcal{A}_0$. In most interesting applications the set $\mathcal{A}_0$ consists of discontinuous functions. The natural space of discontinuous functions is BV space. It turns out that in the general case if G and W are C^1-functions and if we consider

$$\mathcal{A}_{BV} := \mathcal{A}_0 \cap \mathcal{B}(\mathbb{R}^N) \cap BV \cap L^\infty, \tag{1.14}$$

then

$$\mathcal{A}_{BV} \subset \mathcal{A} \subset \mathcal{A}_0. \tag{1.15}$$

In many cases we have $\mathcal{A}_{BV} = \mathcal{A}$. For example this is indeed the case if the energy $I_\varepsilon(v)$ has the simplest form $I_\varepsilon(v) = \int_\Omega \varepsilon|\nabla v|^2 + \frac{1}{\varepsilon}W(v)\,dx$, and the set of zeros of W: $\{h : W(h) = 0\}$ is finite. However, this is in general not the case. For example, as was shown by Ambrosio, De Lellis and Mantegazza in [3], $\mathcal{A}_{BV} \subsetneq \mathcal{A}$ in the particular case of the energy defined by (1.7) with $N = 2$. On the other hand, there are many applications where the set $\mathcal{A}$ still inherits some good properties of BV space. For example, it is indeed the case for the energy (1.7) with $N = 2$, as was shown by Camillo de Lellis and Felix Otto in [23].

Definition 1.2. For every $\boldsymbol{\nu} \in S^{N-1}$ define $Q(\boldsymbol{\nu}) := \big\{y \in \mathbb{R}^N : \ -1/2 < y \cdot \boldsymbol{\nu}_j < 1/2 \quad \forall j\big\}$, where $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N\}$ is an orthonormal base in $\mathbb{R}^N$ such that $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$. Then set

$$\mathcal{D}_1(v^+, v^-, \boldsymbol{\nu}) := \bigg\{v \in C^n\big(\mathbb{R}^N, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) \cap \mathcal{B}(\mathbb{R}^N) : \\ v(y) \equiv \theta(\boldsymbol{\nu} \cdot y) \text{ and } v(y) = v^- \text{ if } y \cdot \boldsymbol{\nu} \leq -1/2, \ v(y) = v^+ \text{ if } y \cdot \boldsymbol{\nu} \geq 1/2\bigg\},$$

where $\mathcal{B}(\cdot)$ is defined in (1.8), and

$$\mathcal{D}_{per}(v^+, v^-, \boldsymbol{\nu}) := \bigg\{v \in C^n\big(\mathbb{R}^N, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) \cap \mathcal{B}(\mathbb{R}^N) : \\ v(y) = v^- \text{ if } y \cdot \boldsymbol{\nu} \leq -1/2, \ v(y) = v^+ \text{ if } y \cdot \boldsymbol{\nu} \geq 1/2, \ v(y + \boldsymbol{\nu}_j) = v(y) \ \forall j = 2, \ldots, N\bigg\}.$$

Next define

$$E_1(v^+, v^-, \boldsymbol{\nu}, x) := \inf\bigg\{\int\limits_{Q(\boldsymbol{\nu}_v)} \frac{1}{L} F\Big(L^n \nabla^n \zeta, \ldots, L\nabla\zeta, \zeta, x\Big) dy : \ L > 0, \, \zeta(y) \in \mathcal{D}_1(v^+, v^-, \boldsymbol{\nu})\bigg\}, \tag{1.16}$$

$$E_{per}(v^+, v^-, \boldsymbol{\nu}, x) := \inf\bigg\{\int\limits_{Q(\boldsymbol{\nu}_v)} \frac{1}{L} F\Big(L^n \nabla^n \zeta, \ldots, L\nabla\zeta, \zeta, x\Big) dy : \ L > 0, \, \zeta(y) \in \mathcal{D}_{per}(v^+, v^-, \boldsymbol{\nu})\bigg\}. \tag{1.17}$$

$$E_{abst}(v^+, v^-, \boldsymbol{\nu}, x) := \Big(\Gamma - \liminf_{\varepsilon \to 0^+} I_\varepsilon\big(Q(\boldsymbol{\nu})\big)\Big)\Big(\eta(v^+, v^-, \boldsymbol{\nu})\Big), \tag{1.18}$$

where

$$\eta(v^+, v^-, \boldsymbol{\nu})(y) := \begin{cases} v^- & \text{if } \boldsymbol{\nu} \cdot y < 0, \\ v^+ & \text{if } \boldsymbol{\nu} \cdot y > 0, \end{cases} \tag{1.19}$$

and we mean the $\Gamma - \liminf$ in L^p topology for some $p \geq 1$.

It is not difficult to deduce that

$$E_{abst}(v^+, v^-, \boldsymbol{\nu}, x) \leq E_{per}(v^+, v^-, \boldsymbol{\nu}, x) \leq E_1(v^+, v^-, \boldsymbol{\nu}, x). \tag{1.20}$$

Next define the functionals $K_1(\cdot), K_{per}(\cdot), K^*(\cdot) : \mathcal{B}(\Omega) \cap BV \cap L^\infty \to \mathbb{R}$ by

$$K_1(v) := \begin{cases} \int_{\Omega \cap J_v} E_1\Big(v^+(x), v^-(x), \boldsymbol{\nu}_v(x), x\Big) \, d\mathcal{H}^{N-1}(x) & \text{if } v \in \mathcal{A}_0, \\ +\infty \quad \text{otherwise}, \end{cases} \tag{1.21}$$

$$K_{per}(v) := \begin{cases} \int_{\Omega \cap J_v} E_{per}\Big(v^+(x), v^-(x), \boldsymbol{\nu}_v(x), x\Big) \, d\mathcal{H}^{N-1}(x) & \text{if } v \in \mathcal{A}_0, \\ +\infty \quad \text{otherwise}, \end{cases} \tag{1.22}$$

$$K^*(v) := \begin{cases} \int_{\Omega \cap J_v} E_{abst}\Big(v^+(x), v^-(x), \boldsymbol{\nu}_v(x), x\Big) \, d\mathcal{H}^{N-1}(x) & \text{if } v \in \mathcal{A}_0, \\ +\infty \quad \text{otherwise}, \end{cases} \tag{1.23}$$

where J_v is the jump set of v, $\boldsymbol{\nu}_v$ is the jump vector and v^-, v^+ are jumps of v. Then, by (1.20) trivially follows

$$K^*(v) \leq K_{per}(v) \leq K_1(v)\,. \tag{1.24}$$

We call $K_1(\cdot)$, $K_{per}(\cdot)$ and $K^*(\cdot)$ by the bound, achieved by one dimensional profiles, multidimensional periodic profiles and abstract profiles respectively.

Our general conjecture is that $K^*(\cdot)$ coincides with the Γ-limit for the functionals $I_\varepsilon(\cdot)$ in (1.10), under L^p convergence, in the case where the limiting functions $v \in BV \cap L^\infty$. It is known that in the case of the problem (1.1), where $W \in C^1$ don't depend on x explicitly, this is indeed the case and moreover, in this case we have equalities in (1.24) (see [2]). The equalities in (1.24) also hold for the functional of the form (1.2), with

$n = 2$, $G(\cdot)/\varepsilon \equiv \varepsilon^3|\nabla^2\psi|^2$ and W which doesn't depend on x explicitly. Moreover, as before, in this case the functional in (1.24) is the Γ-limit (see [15]). The same result is also known for problem (1.7) when $N = 2$ (see [3] and [7],[31]). It is also the case for problem (1.6) where $W(F) = 0$ if and only if $F \in \{A, B\}$, studied by Conti, Fonseca and Leoni, if W satisfies the additional hypothesis (H_3) in [10]. However, as was shown there by an example, if we don't assume (H_3)-hypothesis, then it is possible that $E_{per}\big(\nabla v^+, \nabla v^-, \boldsymbol{\nu}\big)$ is strictly smaller than $E_1\big(\nabla v^+, \nabla v^-, \boldsymbol{\nu}\big)$ and thus, in general, $K_1(\cdot)$ can differ from the Γ-limit. In the same work it was shown that if, instead of (H_3) we assume hypothesis (H_5), then $K_{per}(\cdot)$ turns to be equal to $K^*(\cdot)$ and the Γ-limit of (1.6) equals to $K_{per}(\cdot) \equiv K^*(\cdot)$. The similar result known also for problem (1.2), where $n = 1$ and there exist $\alpha, \beta \in \mathbb{R}^m$ such that $W(h, x) = 0$ if and only if $h \in \{\alpha, \beta\}$, under some restriction on the explicit dependence on x of G and W. As was obtained by Fonseca and Popovici in [16] in this case we also obtain that $K_{per}(\cdot) \equiv K^*(\cdot)$ is the Γ-limit of (1.2). In the case of problem (1.6), where $N = 2$ and $W(QF) = W(F)$ for all $Q \in SO(2)$, Conti and Schweizer in [9] found that the Γ-limit equals to $K^*(\cdot)$ (see also [8] for a related problem). However, by our knowledge, it is not known, whether in general $K^*(\cdot) \equiv K_{per}(\cdot)$.

On [27] we showed that for the general problems (1.2) and (1.4), $K_1(\cdot)$ is the upper bound in the sense of **(**)**, if the limiting function belongs to BV-class. However, as we saw, this bound is not sharp in general. The main result of this paper is that for the general problem (1.10), $K_{per}(\cdot)$ is always an upper bound in the sense of **(**)** in the case where the limiting functions v belong to BV-space and $G, W \in C^1$. More precisely, we have the following Theorem:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and*

$$F : \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N^n} \times \ldots \times \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N} \times \mathbb{R}^{\{k\times N\}+\{d\times N\}+m} \times \mathbb{R}^N \to \mathbb{R}$$

be a nonnegative C^1 function. Furthermore assume that $v := (\nabla u, h, \psi) \in \mathcal{B}(\mathbb{R}^N) \cap BV\big(\mathbb{R}^N, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) \cap L^\infty\big(\mathbb{R}^N, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big)$ satisfies $\operatorname{div} h \equiv 0$, $|Dv|(\partial\Omega) = 0$ and

$$F\Big(0, \ldots, 0, v(x), x\Big) = 0 \quad \textit{for a.e. } x \in \Omega.$$

Then there exists a sequence $v_\varepsilon = \big(\nabla u_\varepsilon, h_\varepsilon, \psi_\varepsilon\big) \in \mathcal{B}(\mathbb{R}^N) \cap C^\infty\big(\mathbb{R}^N, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big)$ such that $\operatorname{div} h_\varepsilon \equiv 0$, for every $p \geq 1$ we have $v_\varepsilon \to v$ in L^p and

$$\lim_{\varepsilon\to 0^+} \int_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon^n \nabla^n v_\varepsilon(x), \ldots, \varepsilon \nabla v_\varepsilon(x)\,,\, v(x)\,,\, x\Big) dx = K_{per}(v).$$

Here $\mathcal{B}(\mathbb{R}^N)$ was defined by (1.8) and $K_{per}(\cdot)$ was defined by (1.22).

For the equivalent formulation and additional details see Theorem 4.2. See also Theorem 3.3 as the analogous result for more general functionals than that defined by (1.5).

Next, as we showed in [34], for the general problem (1.5), when G, W don't depend on x explicitly, $K^*(\cdot)$ is a lower bound in the sense of **(*)**. More precisely, we have the following Theorem:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and*

$$F : \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N^n} \times \ldots \times \mathbb{R}^{\big(\{k\times N\}+\{d\times N\}+m\big)\times N} \times \mathbb{R}^{\{k\times N\}+\{d\times N\}+m} \to \mathbb{R}$$

be a nonnegative continuous function. Furthermore assume that $v := (\nabla u, h, \psi) \in \mathcal{B}(\Omega) \cap BV\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big) \cap L^\infty\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big)$ satisfies

$$F\Big(0, \ldots, 0, v(x)\Big) = 0 \quad \textit{for a.e. } x \in \Omega.$$

Then for every $\{v_\varepsilon\}_{\varepsilon>0} \subset \mathcal{B}(\Omega) \cap W^{n,1}_{loc}\big(\Omega, \mathbb{R}^{k\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^m\big)$, such that $v_\varepsilon \to v$ in L^p as $\varepsilon \to 0^+$, we have

$$\liminf_{\varepsilon\to 0^+} \int_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon^n \nabla^n v_\varepsilon(x), \ldots, \varepsilon \nabla v_\varepsilon(x)\,,\, v(x)\Big) dx \geq K^*(v).$$

Here $K^*(\cdot)$ is defined by (1.23) with respect to L^p topology.

As we saw there is a natural question: whether in general $K^*(\cdot) \equiv K_{per}(\cdot)$? The answer yes will mean that, in the case when G, W are C^1 functions which don't depend on x explicitly, the upper bound in Theorem 1.1 will coincide with the lower bound of Theorem 1.2 and therefore we will find the full Γ-limit in the case of BV limiting functions. The equivalent question is whether

$$E_{abst}(v^+, v^-, \boldsymbol{\nu}, x) = E_{per}(v^+, v^-, \boldsymbol{\nu}, x),$$

where $E_{per}(\cdot)$ is defined in (1.17) and $E_{abst}(\cdot)$ is defined by (1.18). In [34] we formulate and prove some partial results that refer to this important question. In particular we prove that this is indeed the case for the general problem (1.2) i.e. when we have no prescribed differential constraint. More precisely, we have the following Theorem:

Theorem 1.3. *Let $G \in C^1\big(\mathbb{R}^{m\times N^n} \times \mathbb{R}^{m\times N^{(n-1)}} \times \ldots \times \mathbb{R}^{m\times N} \times \mathbb{R}^m, \mathbb{R}\big)$ and $W \in C^1(\mathbb{R}^m, \mathbb{R})$ be nonnegative functions such that $G\big(0, 0, \ldots, 0, b\big) = 0$ for every $b \in \mathbb{R}^m$ and there exist $C > 0$ and $p \geq 1$ satisfying*

$$\frac{1}{C}|A|^p \leq F\Big(A, a_1, \ldots, a_{n-1}, b\Big) \leq C\bigg(|A|^p + \sum_{j=1}^{n-1} |a_j|^p + |b|^p + 1\bigg) \quad \text{for every} \;\; \big(A, a_1, a_2, \ldots, a_{n-1}, b\big), \tag{1.25}$$

where we denote

$$F\Big(A, a_1, \ldots, a_{n-1}, b\Big) := G\Big(A, a_1, \ldots, a_{n-1}, b\Big) + W(b)$$

Next let $\psi \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty$ be such that $\|D\psi\|(\partial\Omega) = 0$ and $W\big(\psi(x)\big) = 0$ for a.e. $x \in \Omega$. Then $K^*(\psi) = K_{per}(\psi)$ and for every $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset W^{n,p}_{loc}(\Omega, \mathbb{R}^m)$ such that $\varphi_\varepsilon \to \psi$ in $L^p_{loc}(\Omega, \mathbb{R}^m)$ as $\varepsilon \to 0^+$, we have

$$\liminf_{\varepsilon\to 0^+} I_\varepsilon(\varphi_\varepsilon) := \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_\Omega F\bigg(\, \varepsilon^n \nabla^n \varphi_\varepsilon(x),\, \varepsilon^{n-1}\nabla^{n-1}\varphi_\varepsilon(x),\, \ldots,\, \varepsilon\nabla\varphi_\varepsilon(x),\, \varphi_\varepsilon(x)\bigg) dx \\ \geq \int_{\Omega\cap J_\psi} \bar{E}_{per}\Big(\psi^+(x), \psi^-(x), \boldsymbol{\nu}(x)\Big) d\mathcal{H}^{N-1}(x)\,, \tag{1.26}$$

where

$$\bar{E}_{per}\Big(\psi^+, \psi^-, \boldsymbol{\nu}\Big) := \\ \inf\bigg\{ \int_{Q_\nu} \frac{1}{L} F\Big(L^n\, \nabla^n\zeta,\, L^{n-1}\, \nabla^{n-1}\zeta,\, \ldots,\, L\,\nabla\zeta,\, \zeta\Big)\, dx : \;\; L \in (0, +\infty)\,,\; \zeta \in \tilde{\mathcal{D}}_{per}(\psi^+, \psi^-, \boldsymbol{\nu})\bigg\}, \tag{1.27}$$

with

$$\tilde{\mathcal{D}}_{per}(\psi^+, \psi^-, \boldsymbol{\nu}) := \bigg\{ \zeta \in C^n(\mathbb{R}^N, \mathbb{R}^m) : \;\; \zeta(y) = \psi^- \;\; \text{if} \;\; y\cdot\boldsymbol{\nu} \leq -1/2, \\ \zeta(y) = \psi^+ \;\; \text{if} \;\; y\cdot\boldsymbol{\nu}(x) \geq 1/2 \;\; \text{and} \;\; \zeta\big(y + \boldsymbol{k}_j\big) = \zeta(y) \;\; \forall j = 2, 3, \ldots, N \bigg\}. \tag{1.28}$$

Here $Q_{\boldsymbol{\nu}} := \{y \in \mathbb{R}^N : \;\; |y\cdot\boldsymbol{\nu}_j| < 1/2 \quad \forall j = 1, 2 \ldots N\}$ where $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_N\} \subset \mathbb{R}^N$ is an orthonormal base in $\mathbb{R}^N$ such that $\boldsymbol{\nu}_1 := \boldsymbol{\nu}$. Moreover, there exists e sequence $\{\psi_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^m)$ such that $\int_\Omega \psi_\varepsilon(x)dx = \int_\Omega \psi(x)dx$, for every $q \geq 1$ we have $\psi_\varepsilon \to \psi$ in $L^q(\Omega, \mathbb{R}^m)$ as $\varepsilon \to 0^+$, and we have

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\psi_\varepsilon) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_\Omega F\bigg(\, \varepsilon^n \nabla^n \psi_\varepsilon(x),\, \varepsilon^{n-1}\nabla^{n-1}\psi_\varepsilon(x),\, \ldots,\, \varepsilon\nabla\psi_\varepsilon(x),\, \psi_\varepsilon(x)\bigg) dx \\ = \int_{\Omega\cap J_\psi} \bar{E}_{per}\Big(\psi^+(x), \psi^-(x), \boldsymbol{\nu}(x)\Big) d\mathcal{H}^{N-1}(x)\,. \tag{1.29}$$

Remark 1.1. In what follows we use some special notations and apply some basic theorems about BV functions. For the convenience of the reader we put these notations and theorems in Appendix.

2 Preliminary results

Definition 2.1. Let $\boldsymbol{T} \in \mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$. For any $A_1, A_2 \in \big\{A \in \mathbb{R}^m : \exists M \in \mathbb{R}^{d\times N},\ A = \boldsymbol{T}\cdot M\big\}$, for every $\boldsymbol{\nu} \in \mathbb{R}^N$ satisfying $|\boldsymbol{\nu}| = 1$ and for every system of $N-1$ linearly independent vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\}$ in $\mathbb{R}^N$ satisfying $\boldsymbol{a}_j \cdot \boldsymbol{\nu} = 0$ for all $j = 1, 2, \dots, (N-1)$ set

$$\mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\big) := \Big\{u \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) :\ \boldsymbol{T}\cdot\nabla u(y) = A_1 \ \text{ if } \ y\cdot\boldsymbol{\nu} \leq -1/2,$$
$$\boldsymbol{T}\cdot\nabla u(y) = A_2 \ \text{ if } \ y\cdot\boldsymbol{\nu} \geq 1/2 \ \text{ and } \ \boldsymbol{T}\cdot\nabla u(y+\boldsymbol{a}_j) = \boldsymbol{T}\cdot\nabla u(y) \ \ \forall j = 1, 2, \dots, (N-1)\Big\}, \quad (2.1)$$

and

$$\tilde{\mathcal{D}}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\big) := \Big\{u \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^d) :$$
$$\boldsymbol{T}\cdot\nabla u \in C^1(\mathbb{R}^N, \mathbb{R}^m),\ \ \boldsymbol{T}\cdot\nabla u(y) = A_1 \ \text{ if } \ y\cdot\boldsymbol{\nu} \leq -1/2,$$
$$\boldsymbol{T}\cdot\nabla u(y) = A_2 \ \text{ if } \ y\cdot\boldsymbol{\nu} \geq 1/2 \ \text{ and } \ \boldsymbol{T}\cdot\nabla u(y+\boldsymbol{a}_j) = \boldsymbol{T}\cdot\nabla u(y) \ \ \forall j = 1, 2, \dots, (N-1)\Big\}. \quad (2.2)$$

Set also

$$\mathcal{D}_0\big(\boldsymbol{\nu}, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\big) := \Big\{u \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) :\ u(y) = 0 \ \text{ if } \ |y\cdot\boldsymbol{\nu}| \geq 1/2$$
$$\text{and } \ u(y+\boldsymbol{a}_j) = u(y) \ \ \forall j = 1, 2, \dots, (N-1)\Big\}. \quad (2.3)$$

and

$$\mathcal{D}_1\big(\boldsymbol{T}, \boldsymbol{\nu}, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\big) :=$$
$$\Big\{u \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap Lip(\mathbb{R}^N, \mathbb{R}^d) :\ \boldsymbol{T}\cdot\nabla u(y) = 0 \ \text{ if } \ |y\cdot\boldsymbol{\nu}| \geq 1/2$$
$$\text{and } \ u(y+\boldsymbol{a}_j) = u(y) \ \ \forall j = 1, 2, \dots, (N-1)\Big\}. \quad (2.4)$$

Proposition 2.1. *Let $\boldsymbol{T} \in \mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$ and $G \in C^1(\mathbb{R}^{m\times N} \times \mathbb{R}^m \times \mathbb{R})$, satisfying $G \geq 0$, and let $A_1, A_2 \in \big\{A \in \mathbb{R}^m : \exists M \in \mathbb{R}^{d\times N},\ A = \boldsymbol{T}\cdot M\big\}$ satisfy $G(0, A_1, -1) = 0$ and $G(0, A_2, 1) = 0$. Furthermore, let $\boldsymbol{\nu} \in \mathbb{R}^N$ satisfies $|\boldsymbol{\nu}| = 1$. Given a system $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_{N-1}\}$ of $(N-1)$ linearly independent vectors in $\mathbb{R}^N$, satisfying $\boldsymbol{p}_j \cdot \boldsymbol{\nu} = 0$ for all $j = 1, 2, \dots, (N-1)$, set*

$$\Theta(\boldsymbol{p}_1, \dots, \boldsymbol{p}_{N-1}) :=$$
$$\inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}\boldsymbol{p}_j\Big),\ \boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}\boldsymbol{p}_j\Big),\ s_1/|s_1|\bigg)\,ds :$$
$$L > 0,\ u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{N-1}\big)\Bigg\} \qquad \textit{(see Definition 2.1)},$$

where

$$I_N := \Big\{s \in \mathbb{R}^N :\ -1/2 < s_j < 1/2 \quad \forall j = 1, 2, \dots, N\Big\}. \quad (2.5)$$

Then for every two systems $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{N-1}\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{N-1}\}$ of $(N-1)$ linearly independent vectors in $\mathbb{R}^N$, satisfying $\boldsymbol{a}_j \cdot \boldsymbol{\nu} = 0$ and $\boldsymbol{b}_j \cdot \boldsymbol{\nu} = 0$ for all $j = 1, 2, \dots, (N-1)$, we have

$$\Theta(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{N-1}) = \Theta(\boldsymbol{b}_1, \dots, \boldsymbol{b}_{N-1}). \quad (2.6)$$

Proof. First of all we observe that if $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{N-1}\}$ are equal, up to a permutation of vectors, then (2.6) will follow by the definition. Next since for every $(N-1)$ positive integer numbers $K_1, \ldots, K_{N-1}$ we have $\mathcal{D}(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}) \subset \mathcal{D}(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, K_1\boldsymbol{a}_1, \ldots, K_{N-1}\boldsymbol{a}_{N-1})$ then

$$\begin{gathered}\Theta(K_1\boldsymbol{a}_1, \ldots, K_{N-1}\boldsymbol{a}_{N-1}) = \\ \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K_j\boldsymbol{a}_j\Big),\, \boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K_j\boldsymbol{a}_j\Big),\, s_1/|s_1|\bigg)\,ds : \\ L > 0,\, u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, K_1\boldsymbol{a}_1, \ldots, K_{N-1}\boldsymbol{a}_{N-1}\big)\Bigg\} \le \\ \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K_j\boldsymbol{a}_j\Big),\, \boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K_j\boldsymbol{a}_j\Big),\, s_1/|s_1|\bigg)\,ds : \\ L > 0,\, u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)\Bigg\}. \end{gathered} \tag{2.7}$$

Thus changing variables of the integration $\bar{s}_1 = s_1$ and $\bar{s}_j = K_{j-1}s_j$ for every $j = 2, \ldots, N$ in the r.h.s. of (2.7) and using the periodicity condition in the definition of $\mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)$ gives

$$\Theta(K_1\boldsymbol{a}_1, \ldots, K_{N-1}\boldsymbol{a}_{N-1}) \le \Theta(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1})\,. \tag{2.8}$$

On the other hand for every positive integer number K we have that if $u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_{N-1}\big)$ then the function $\bar{u}(y) := \frac{1}{K}u(Ky) \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)$ and therefore,

$$\begin{gathered}\Theta(K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_{N-1}) = \\ \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K\boldsymbol{a}_j\Big),\, \boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K\boldsymbol{a}_j\Big),\, s_1/|s_1|\bigg)\,ds : \\ L > 0,\, u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_{N-1}\big)\Bigg\} \ge \\ \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K\boldsymbol{a}_j\Big),\, \boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}K\boldsymbol{a}_j\Big),\, s_1/|s_1|\bigg)\,ds : \\ L > 0,\, \frac{1}{K}u(Ky) \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)\Bigg\}. \end{gathered} \tag{2.9}$$

Therefore, changing variables $\bar{s}_1 := s_1/K$, $\bar{s}_j = s_j$ for $j = 2, \ldots, N$ in the r.h.s. of (2.9) we obtain

$$\begin{gathered}\Theta(K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_{N-1}) \ge \\ \inf\Bigg\{\frac{K}{L}\int\limits_{I_N} G\bigg((L/K)\,\nabla\{\boldsymbol{T}\cdot\nabla \bar{u}\}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}\boldsymbol{a}_j\Big),\, \boldsymbol{T}\cdot\nabla \bar{u}\Big(s_1\boldsymbol{\nu} + \Sigma_{j=1}^{N-1} s_{j+1}\boldsymbol{a}_j\Big),\, s_1/|s_1|\bigg)\,ds : \\ L > 0,\, \bar{u} \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)\Bigg\} = \Theta(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1})\,. \end{gathered} \tag{2.10}$$

Plugging (2.10) to (2.8) with $K_j := K$ we deduce

$$\Theta(K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_{N-1}) = \Theta(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1})\,. \tag{2.11}$$

Thus from (2.8) and (2.11) we deduce

$$\Theta\big((K_1/K)\boldsymbol{a}_1, \ldots, (K_{N-1}/K)\boldsymbol{a}_{N-1}\big) \le \Theta(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1})\,. \tag{2.12}$$

So for every $(N-1)$ positive rational numbers $r_1,\ldots,r_{N-1}$ we have

$$\Theta(r_1\boldsymbol{a}_1,\ldots,r_{N-1}\boldsymbol{a}_{N-1})\leq\Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})\,. \tag{2.13}$$

Then also

$$\Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})=\Theta\big((1/r_1)r_1\boldsymbol{a}_1,\ldots,(1/r_{N-1})r_{N-1}\boldsymbol{a}_{N-1}\big)\leq\Theta(r_1\boldsymbol{a}_1,\ldots,r_{N-1}\boldsymbol{a}_{N-1})\,. \tag{2.14}$$

Therefore for every $(N-1)$ positive rational numbers $r_1,\ldots,r_{N-1}$ we must have

$$\Theta(r_1\boldsymbol{a}_1,\ldots,r_{N-1}\boldsymbol{a}_{N-1})=\Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})\,. \tag{2.15}$$

Finally since

$$\mathcal{D}\big(A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_j\ldots,\boldsymbol{a}_{N-1}\big)=\mathcal{D}\big(A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,,\ldots,(-\boldsymbol{a}_j)\ldots,\boldsymbol{a}_{N-1}\big)\,,$$

we deduce that the equality

$$\Theta(r_1\boldsymbol{a}_1,\ldots,r_{N-1}\boldsymbol{a}_{N-1})=\Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}) \tag{2.16}$$

is valid for every $(N-1)$ different from zero rational numbers $r_1,\ldots,r_{N-1}$ (without any restriction on their sign).

Next since obviously

$$\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_{N-1}\big)=\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,(\boldsymbol{a}_1+\boldsymbol{a}_2),\boldsymbol{a}_3,\ldots,\boldsymbol{a}_{N-1}\big)$$

we have

$$\Theta\big(\boldsymbol{a}_1,(\boldsymbol{a}_1+\boldsymbol{a}_2),\boldsymbol{a}_3,\ldots,\boldsymbol{a}_{N-1}\big)=\inf\Bigg\{\frac{1}{L}\int\limits_{I_N}\times$$
$$G\bigg(L\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu}+(s_2+s_3)\boldsymbol{a}_1+\Sigma_{j=2}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu}+(s_2+s_3)\boldsymbol{a}_1+\Sigma_{j=2}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),s_1/|s_1|\bigg)$$
$$\times ds:\qquad L>0,\;u\in\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_{N-1}\big)\Bigg\}\,. \tag{2.17}$$

Therefore, changing variables $\bar{s}_j=s_j$ if $j\neq 2$ and $\bar{s}_2=s_2+s_3$ in the r.h.s. of (2.17) and using the periodicity condition of the Definition 2.1 gives

$$\Theta\big(\boldsymbol{a}_1,(\boldsymbol{a}_1+\boldsymbol{a}_2),\boldsymbol{a}_3,\ldots,\boldsymbol{a}_{N-1}\big)=\Theta(\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_{N-1})\,. \tag{2.18}$$

Next let $\boldsymbol{b}_j=\Sigma_{j=1}^{N-1}Q_{jk}\boldsymbol{a}_k$, where $\big\{Q_{jk}\big\}\in\mathbb{R}^{(N-1)\times(N-1)}$ is a non-degenerate matrix with rational coefficients. Then we can obtain the system $\{\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_{N-1}\}$ from the system $\{\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_{N-1}\}$ step by step by applying the following three types of operations

- Multiplying the vectors of the system with different from zero rational numbers.
- Permutation of the vectors of the system.
- Adding the first vector of the system to the second one.

Since by (2.16) and (2.18) every step keeps the same $\Theta(\cdot,\ldots,\cdot)$ we deduce that the equality

$$\Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})=\Theta(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1})\,, \tag{2.19}$$

is valid for every $\boldsymbol{b}_j=\Sigma_{j=1}^{N-1}Q_{jk}\boldsymbol{a}_k$, where $\big\{Q_{jk}\big\}$ is a non-degenerate matrix with rational coefficients.

Finally consider the general situation where $\boldsymbol{b}_j = \Sigma_{j=1}^{N-1} R_{jk}\boldsymbol{a}_k$, where $\{R_{jk}\} \in \mathbb{R}^{(N-1)\times(N-1)}$ is a non-degenerate matrix with real coefficients. Let $\delta > 0$ be an arbitrary small positive number. By the definition of $\Theta(\cdot,\ldots,\cdot)$ there exists $L_\delta > 0$ and $u_\delta \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}\big)$ such that

$$\frac{1}{L_\delta}\int\limits_{I_N} G\Big(L_\delta\,\nabla\{\boldsymbol{T}\cdot\nabla u_\delta\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u_\delta\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\Big)\,ds$$
$$< \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}) + \frac{\delta}{2}\,. \tag{2.20}$$

There exists a sequence of non-degenerate matrices with rational coefficients $\big\{\{Q^{(n)}_{jk}\}\big\}_{n=1}^{\infty} \subset \mathbb{R}^{(N-1)\times(N-1)}$ with the property that $\lim_{n\to\infty}\{Q^{(n)}_{jk}\} = \{R_{jk}\}$ and $\lim_{n\to\infty}\{Q^{(n)}_{jk}\}^{-1} = \{R_{jk}\}^{-1}$. In particular if we set $\{P^{(n)}_{jk}\} := \{Q^{(n)}_{jk}\}^{-1}\cdot\{R_{jk}\}$ then $\lim_{n\to\infty}\{P^{(n)}_{jk}\} = I_{N-1}$ where $I_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix. For every n consider $(N-1)$ linearly independent vectors $\boldsymbol{c}^{(n)}_1,\ldots,\boldsymbol{c}^{(n)}_{N-1}$ satisfying $\boldsymbol{c}^{(n)}_j = \Sigma_{k=1}^{N-1}P^{(n)}_{jk}\boldsymbol{a}_k$ for all j. Thus $\boldsymbol{b}_j = \Sigma_{k=1}^{N-1}Q^{(n)}_{jk}\boldsymbol{c}^{(n)}_k$. Next let $u_n(y) : \mathbb{R}^N \to \mathbb{R}^d$ be defined by

$$u_n\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{c}^{(n)}_j\Big) = u_\delta\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big)\qquad \forall s=(s_1,s_2,\ldots,s_N)\in\mathbb{R}^N\,.$$

Thus by (2.20) and by the equality $\lim_{n\to\infty}\{P^{(n)}_{jk}\} = I_{N-1}$, for sufficiently large n we must have

$$\frac{1}{L_\delta}\int\limits_{I_N} G\Big(L_\delta\,\nabla\{\boldsymbol{T}\cdot\nabla u_n\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{c}^{(n)}_j\Big),\,\boldsymbol{T}\cdot\nabla u_n\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{c}^{(n)}_j\Big)\Big)\,ds$$
$$< \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}) + \delta\,. \tag{2.21}$$

However, we have $u_n \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{c}^{(n)}_1, \boldsymbol{c}^{(n)}_2, \ldots, \boldsymbol{c}^{(n)}_{N-1}\big)$. Thus by (2.21) we obtain

$$\Theta(\boldsymbol{c}^{(n)}_1,\ldots,\boldsymbol{c}^{(n)}_{N-1}) < \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}) + \delta\,. \tag{2.22}$$

On the other hand, by (2.19) and the equality $\boldsymbol{b}_j = \Sigma_{k=1}^{N-1}Q^{(n)}_{jk}\boldsymbol{c}^{(n)}_k$ we have

$$\Theta(c^{(n)}_1,\ldots,\boldsymbol{c}^{(n)}_{N-1}) = \Theta(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1})\,.$$

Therefore, by (2.22)

$$\Theta(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1}) < \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}) + \delta\,, \tag{2.23}$$

and since $\delta > 0$ was arbitrary small we finally get

$$\Theta(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1}) \le \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})\,. \tag{2.24}$$

Interchanging the roles of $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\}$ and $\{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1}\}$ we obtain the opposite inequality. So in fact we have the equality

$$\Theta(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{N-1}) = \Theta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1})$$

and the result follows. $\square$

Lemma 2.1. *Let $\boldsymbol{T} \in \mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$ and let $G \in C^1(\mathbb{R}^{m\times N}\times\mathbb{R}^m\times\mathbb{R})$, satisfying $G \ge 0$, and let $A_1, A_2, \boldsymbol{\nu}$, $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}\}$ be the same as in Definition 2.1 and satisfy $G(0, A_1, -1) = 0$ and $G(0, A_2, 1) = 0$. Furthermore, set*

$$R_L := \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\Big(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\Big)\,ds :$$
$$u \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)\Bigg\}\,, \tag{2.25}$$

and

$$\tilde{R}_L := \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\Big(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\Big)\,ds :$$

$$u\in\tilde{\mathcal{D}}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\Bigg\} \tag{2.26}$$

(see Definition 2.1). Then

$$R_L=\tilde{R}_L \qquad \forall L>0\,. \tag{2.27}$$

and

$$\inf_{L>0}R_L=\inf_{L\in(0,1)}R_L=\lim_{L\to0^+}R_L\,. \tag{2.28}$$

Proof. Firstly we will prove (2.27). By Definition 2.1 we clearly have

$$\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\subset\tilde{\mathcal{D}}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big).$$

Therefore for every $L>0$ we clearly deduce $R_L\geq\tilde{R}_L$. Next fix an arbitrary $u\in\tilde{\mathcal{D}}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ and consider $\zeta(z)\in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ such that $\operatorname{supp}\zeta\subset\subset B_1(0)$, $\zeta\geq0$ and $\int_{\mathbb{R}^N}\zeta(z)dz=1$. For any $\varepsilon>0$ and any fixed $x\in\mathbb{R}^N$ set

$$\bar{u}_\varepsilon(x):=\frac{1}{\varepsilon^N}\Big\langle\zeta\Big(\frac{y-x}{\varepsilon}\Big),u(y)\Big\rangle \tag{2.29}$$

(see notations and definitions in the Appendix). Then $\bar{u}_\varepsilon\in C^\infty(\mathbb{R}^N,\mathbb{R}^d)$. Moreover, clearly

$$\boldsymbol{T}\cdot\{\nabla\bar{u}_\varepsilon(x)\}=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\zeta\Big(\frac{y-x}{\varepsilon}\Big)\cdot\{\boldsymbol{T}\cdot\nabla u\}(y)dy=\int_{\mathbb{R}^N}\zeta(z)\cdot\big\{\boldsymbol{T}\cdot\nabla u\big\}(x+\varepsilon z)dz, \tag{2.30}$$

and

$$\nabla\big(\boldsymbol{T}\cdot\{\nabla\bar{u}_\varepsilon(x)\}\big)=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\zeta\Big(\frac{y-x}{\varepsilon}\Big)\cdot\nabla\{\boldsymbol{T}\cdot\nabla u\}(y)dy=\int_{\mathbb{R}^N}\zeta(z)\cdot\nabla\big\{\boldsymbol{T}\cdot\nabla u\big\}(x+\varepsilon z)dz, \tag{2.31}$$

Then by the definition of $\tilde{\mathcal{D}}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ and by (2.30) we have

$$\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon(y)=A_1\ \text{ if }\ y\cdot\boldsymbol{\nu}\leq-(1/2+\varepsilon),\ \boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon(y)=A_2\ \text{ if }\ y\cdot\boldsymbol{\nu}\geq(1/2+\varepsilon)\ \text{ and}$$
$$\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon(y+\boldsymbol{a}_j)=\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon(y)\ \ \forall j=1,2,\ldots,(N-1)\,. \tag{2.32}$$

Moreover, $\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon\to\boldsymbol{T}\cdot\nabla u$ as $\varepsilon\to0^+$ in $C^1(\mathbb{R}^N,\mathbb{R}^m)$ i.e. $\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon\to\boldsymbol{T}\cdot\nabla u$ uniformly in $\mathbb{R}^N$ and $\nabla\{\boldsymbol{T}\cdot\nabla\bar{u}_\varepsilon\}\to\nabla\{\boldsymbol{T}\cdot\nabla u\}$ uniformly in $\mathbb{R}^N$. Finally define $u_\varepsilon\in C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ by the formula

$$u_\varepsilon\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big):=\bar{u}_\varepsilon\Big((2\varepsilon+1)s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big)\,. \tag{2.33}$$

Then using (2.32) we deduce that $u_\varepsilon\in\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$. Finally, $\boldsymbol{T}\cdot\nabla u_\varepsilon\to\boldsymbol{T}\cdot\nabla u$ as $\varepsilon\to0^+$ in $C^1(\mathbb{R}^N,\mathbb{R}^m)$. Therefore,

$$R_L\leq\lim_{\varepsilon\to0^+}\Bigg\{\frac{1}{L}\int\limits_{I_N}G\Big(L\,\nabla\{\boldsymbol{T}\cdot\nabla u_\varepsilon\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u_\varepsilon\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\Big)\,ds\Bigg\}$$
$$=\frac{1}{L}\int\limits_{I_N}G\Big(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\Big)\,ds\,. \tag{2.34}$$

Thus since $u\in\tilde{\mathcal{D}}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ was chosen arbitrary, from (2.34) we deduce $R_L\leq\tilde{R}_L$, which together with the reverse inequality, established before, gives (2.27).

We are going to prove (2.28) now. For every positive integer number K we have that if

$$u\in\mathcal{D}\big(\boldsymbol{T},A_2,A_1,\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$$

then the function

$$w_K(y) := \frac{1}{K}u(Ky) \in \mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)\,, \tag{2.35}$$

Moreover, changing variables in the integration and using the periodicity conditions of the definition of

$$\mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)$$

gives the following equality

$$\begin{aligned}\frac{K}{L}\int\limits_{I_N} G\bigg((L/K)\,\nabla\{\boldsymbol{T}\cdot\nabla w_K\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla w_K\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\bigg)\,ds\\ =\frac{1}{L}\int\limits_{I_N} G\bigg(L\,\nabla\{\boldsymbol{T}\cdot\nabla u\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\bigg)\,ds\,.\end{aligned} \tag{2.36}$$

Then in particular we observe that, for $L>0$ and a positive integer K, we have

$$R_L \le R_{KL}\,. \tag{2.37}$$

and thus clearly we have

$$\inf_{L>0} R_L = \inf_{L\in(0,1)} R_L = \liminf_{L\to 0^+} R_L\,. \tag{2.38}$$

Finally assume, by the contradiction, that $\limsup\limits_{L\to 0^+} R_L > \liminf\limits_{L\to 0^+} R_L$. Then there exists $\delta>0$ and a sequence $L_n\to 0^+$ as $n\to+\infty$ such that

$$\lim_{n\to+\infty} R_{L_n} > \inf_{L>0} R_L + 2\delta\,.$$

Thus in particular, there exists $L_0>0$ and $u_0\in\mathcal{D}\big(\boldsymbol{T}, A_2, A_1, \boldsymbol{\nu}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}\big)$ such that

$$\lim_{n\to+\infty} R_{L_n} > \delta+\frac{1}{L_0}\int\limits_{I_N} G\bigg(L_0\nabla\{\boldsymbol{T}\cdot\nabla u_0\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\boldsymbol{T}\cdot\nabla u_0\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),s_1/|s_1|\bigg)ds. \tag{2.39}$$

Set

$$K_n := \max\big\{K\in\mathbb{N} : K\le (L_0/L_n)\big\}\,. \tag{2.40}$$

Then by the definition $K_n \le (L_0/L_n) < K_n+1$ and $\lim_{n\to+\infty} K_n = +\infty$. Thus if we set $d_n := L_n\cdot K_n$ then $\lim_{n\to+\infty} d_n = L_0$. On the other hand by (2.37) and (2.39) we obtain

$$\begin{aligned}\liminf_{n\to+\infty} R_{d_n} := \liminf_{n\to+\infty} R_{K_nL_n} > \delta+\\ \frac{1}{L_0}\int\limits_{I_N} G\bigg(L_0\,\nabla\{\boldsymbol{T}\cdot\nabla u_0\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u_0\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\bigg)\,ds\,.\end{aligned} \tag{2.41}$$

Thus in particular by the definition of R_{d_n}

$$\begin{aligned}\liminf_{n\to+\infty}\Bigg\{\frac{1}{d_n}\int\limits_{I_N} G\bigg(d_n\,\nabla\{\boldsymbol{T}\cdot\nabla u_0\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u_0\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\bigg)\,ds\Bigg\} >\\ \delta+\frac{1}{L_0}\int\limits_{I_N} G\bigg(L_0\,\nabla\{\boldsymbol{T}\cdot\nabla u_0\}\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\boldsymbol{T}\cdot\nabla u_0\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,s_1/|s_1|\bigg)\,ds\,,\end{aligned} \tag{2.42}$$

which contradicts to the identity $\lim_{n\to+\infty} d_n = L_0$. So we have (2.28). $\square$

Lemma 2.2. *Let $\boldsymbol{T}\in\mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$ and let $G\in C^1(\mathbb{R}^{m\times N}\times\mathbb{R}^m\times\mathbb{R})$, satisfying $G\ge 0$, and let $A_1, A_2, \boldsymbol{\nu}$, $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{N-1}\}$ be the same as in Definition 2.1 and satisfy $G(0, A_1, -1) = 0$ and $G(0, A_2, 1) = 0$. Let $\theta(t)\in C^1(\mathbb{R},\mathbb{R})$ satisfies $\theta(t) = 0$ if $t\le -1/2$ and $\theta(t) = 1$ if $t\ge 1/2$. For every $L\in(0,1)$ define the function $m_L : \mathbb{R}^N\to\mathbb{R}^m$ by*

$$m_L(y) := \big(1-\theta(y\cdot\boldsymbol{\nu}/L)\big)A_1 + \theta(y\cdot\boldsymbol{\nu}/L)A_2\,. \tag{2.43}$$

Then

$$\inf_{L\in(0,1)} \tilde{P}_L = \lim_{L\to 0^+} P_L\,, \tag{2.44}$$

where

$$P_L := \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\Big(L\nabla(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\Big)ds:\\ v\in\mathcal{D}_0\big(\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\Bigg\}, \tag{2.45}$$

and

$$\tilde{P}_L := \inf\Bigg\{\frac{1}{L}\int\limits_{I_N} G\Big(L\nabla(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\Big)ds:\\ v\in\mathcal{D}_1\big(\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\Bigg\}, \tag{2.46}$$

with $\mathcal{D}_1(\cdot)$ and $\mathcal{D}_0(\cdot)$ defined by Definition 2.1.

Proof. Since $\mathcal{D}_0\big(\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\subset\mathcal{D}_1\big(\boldsymbol{T},\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ we deduce

$$\inf_{L\in(0,1)} \tilde{P}_L \leq \liminf_{L\to 0^+} P_L\,. \tag{2.47}$$

Next let $L\in(0,1)$ and $v\in\mathcal{D}_1\big(\boldsymbol{T},\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ that we now fix. As before for every positive integer $K\geq 2$ define

$$v_K(y) := \frac{1}{K}v(Ky)\in\mathcal{D}_1\big(\boldsymbol{T},\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\,. \tag{2.48}$$

Then, changing variables in the integration and using the periodicity condition in the definition of $\mathcal{D}_1(\cdot)$ together with the definition of m_L, we obtain

$$\frac{K}{L}\int\limits_{I_N} G\Big(\frac{L}{K}\nabla(m_{L/K}+\boldsymbol{T}\cdot\nabla v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_{L/K}+\boldsymbol{T}\cdot\nabla v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\Big)ds\\ =\frac{1}{L}\int\limits_{I_N} G\Big(L\,\nabla(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\,\frac{s_1}{|s_1|}\Big)\,ds\,. \tag{2.49}$$

Next let $\zeta(t)\in C_c^\infty(\mathbb{R},\mathbb{R})$ satisfies $\operatorname{supp}\zeta\subset[-1/2,1/2]$, $\zeta(t)=1$ if $t\in[-1/4,1/4]$ and $0\leq\zeta(t)\leq 1$ for every $t\in\mathbb{R}$. For every integer $K\geq 2$ set

$$\tilde{v}_K(y) := v_K(y)\zeta(y\cdot\boldsymbol{\nu})\in\mathcal{D}_0\big(\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)\,. \tag{2.50}$$

Then for $K\geq 2$, we have

$$\boldsymbol{T}\cdot\nabla\tilde{v}_K(y)=\boldsymbol{T}\cdot\Big(\nabla v_K(y)\zeta(y\cdot\boldsymbol{\nu})+\zeta'(y\cdot\boldsymbol{\nu})v_K(y)\otimes\boldsymbol{\nu}\Big)=\boldsymbol{T}\cdot\nabla v(Ky)+\frac{1}{K}\zeta'(y\cdot\boldsymbol{\nu})\boldsymbol{T}\cdot\big\{v(Ky)\otimes\boldsymbol{\nu}\big\}\,, \tag{2.51}$$

and furthermore, for the same case $K\geq 2$,

$$\nabla\{\boldsymbol{T}\cdot\nabla\tilde{v}_K\}(y)=\nabla\{\boldsymbol{T}\cdot\nabla v_K\}(y)+\zeta'(y\cdot\boldsymbol{\nu})\nabla\Big(\boldsymbol{T}\cdot\{v_K(y)\otimes\boldsymbol{\nu}\}\Big)+\zeta''(y\cdot\boldsymbol{\nu})\Big(\boldsymbol{T}\cdot\{v_K(y)\otimes\boldsymbol{\nu}\}\Big)\otimes\boldsymbol{\nu}\\ =K\nabla\{\boldsymbol{T}\cdot\nabla v\}(Ky)+\zeta'(y\cdot\boldsymbol{\nu})\nabla\Big(\boldsymbol{T}\cdot\{v\otimes\boldsymbol{\nu}\}\Big)(Ky)+\frac{1}{K}\zeta''(y\cdot\boldsymbol{\nu})\Big(\boldsymbol{T}\cdot\{v(Ky)\otimes\boldsymbol{\nu}\}\Big)\otimes\boldsymbol{\nu}\,. \tag{2.52}$$

Thus

$$(L/K)\nabla\{\boldsymbol{T}\cdot\nabla\tilde{v}_K\}(y)=\\ L\nabla\{\boldsymbol{T}\cdot\nabla v\}(Ky)+\frac{L}{K}\zeta'(y\cdot\boldsymbol{\nu})\nabla\big(\boldsymbol{T}\cdot\{v\otimes\boldsymbol{\nu}\}\big)(Ky)+\frac{L}{K^2}\zeta''(y\cdot\boldsymbol{\nu})\Big(\boldsymbol{T}\cdot\{v(Ky)\otimes\boldsymbol{\nu}\}\Big)\otimes\boldsymbol{\nu}\,. \tag{2.53}$$

On the other hand by (2.43) we have

$$\begin{aligned} m_{L/K}(y) &= \big(1-\theta(y\cdot\boldsymbol{\nu}\,K/L)\big)A_1+\theta(y\cdot\boldsymbol{\nu}\,K/L)A_2 \quad \forall y\in\mathbb{R}^N\,,\\ (L/K)\nabla m_{L/K}(y) &= \theta'(y\cdot\boldsymbol{\nu}\,K/L)\,(A_2-A_1)\otimes\boldsymbol{\nu} \quad \forall y\in\mathbb{R}^N\,. \end{aligned} \tag{2.54}$$

However, we have $\zeta,\zeta',\zeta'',\theta,\theta'\in L^\infty$ and $v,\nabla v,\nabla\{\boldsymbol{T}\cdot\nabla v\}\in L^\infty$. Moreover, $G\big(0,A_r,(-1)^r\big)=0$ and since $G\geq 0$ also $\nabla G\big(0,A_r,(-1)^r\big)=0$ for every $r\in\{1,2\}$. Therefore, by (2.54), (2.51) and (2.53) we have

$$\begin{aligned} &\frac{K}{L}\int_{I_N} G\bigg(\frac{L}{K}\nabla(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)ds\\ &-\frac{K}{L}\int_{I_N} G\bigg(\frac{L}{K}\nabla(m_{L/K}+\boldsymbol{T}\cdot\nabla v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_{L/K}+\boldsymbol{T}\cdot\nabla v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)ds\\ &\qquad\to 0 \qquad\qquad \text{as}\quad K\to+\infty\,. \end{aligned} \tag{2.55}$$

Thus by (2.49) we have

$$\begin{aligned} &\lim_{K\to+\infty}\frac{K}{L}\int_{I_N}\Bigg\{\\ &\quad G\bigg(\frac{L}{K}\nabla(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)\Bigg\}ds\\ &\quad=\frac{1}{L}\int_{I_N} G\bigg(L\nabla(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)ds\,. \end{aligned} \tag{2.56}$$

However, since $\tilde v_K(y)\in\mathcal{D}_0\big(\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$, we obtain

$$\begin{aligned} &\lim_{K\to+\infty}\Bigg\{\\ &\frac{K}{L}\int_{I_N} G\bigg(\frac{L}{K}\nabla(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_{L/K}+\boldsymbol{T}\cdot\nabla\tilde v_K)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)ds\\ &\qquad\qquad\qquad\Bigg\}\geq\limsup_{l\to0^+}P_l \end{aligned} \tag{2.57}$$

Therefore, by (2.56) and (2.57) we deduce

$$\frac{1}{L}\int_{I_N} G\bigg(L\nabla(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),(m_L+\boldsymbol{T}\cdot\nabla v)\Big(s_1\boldsymbol{\nu}+\Sigma_{j=1}^{N-1}s_{j+1}\boldsymbol{a}_j\Big),\frac{s_1}{|s_1|}\bigg)ds\geq\limsup_{l\to0^+}P_l \tag{2.58}$$

Then since $L\in(0,1)$ and $v\in\mathcal{D}_1\big(\boldsymbol{T},\boldsymbol{\nu},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{N-1}\big)$ were arbitrary by (2.58) we deduce

$$\inf_{L\in(0,1)}\tilde P_L\geq\limsup_{L\to0^+}P_L$$

This inequality together with the reverse inequality (2.47) gives equality (2.44). $\square$

3 Upper bound construction

3.1 Primary approximating sequence

We define a special class of mollifiers that we shall use in the upper bound construction.

Definition 3.1. The class $\mathcal{V}_0$ consists of all functions $\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that η is radial, $\eta \geq 0$, $\operatorname{supp} \eta \subset \bar{B}_{1/2}(0)$ and $\int_{\mathbb{R}^N} \eta(z)dz = 1$.

Next let $\eta(z) \in \mathcal{V}_0$, $\boldsymbol{A} \in \mathcal{L}(\mathbb{R}^{d\times N}; \mathbb{R}^m)$, $\boldsymbol{B} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^k)$ and $v \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^d)$ such that $\boldsymbol{A}\cdot\nabla v \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty$ and $\boldsymbol{B} \cdot v \in W^{1,1}(\mathbb{R}^N, \mathbb{R}^k) \cap L^\infty \cap Lip$. For any $\varepsilon > 0$ and any fixed $x \in \mathbb{R}^N$ set

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\Big\langle \eta\Big(\frac{y-x}{\varepsilon}\Big), v(y)\Big\rangle \tag{3.1}$$

(see notations and definitions in the Appendix). Then $\psi_\varepsilon \in C^\infty(\mathbb{R}^N, \mathbb{R}^d)$. Moreover clearly

$$\boldsymbol{A} \cdot \{\nabla\psi_\varepsilon(x)\} = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big) \cdot \{\boldsymbol{A} \cdot \nabla v\}(y)dy = \int_{\mathbb{R}^N} \eta(z) \cdot \big\{\boldsymbol{A} \cdot \nabla v\big\}(x+\varepsilon z)dz, \tag{3.2}$$

$$\boldsymbol{B} \cdot \{\psi_\varepsilon(x)\} = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big) \cdot \{\boldsymbol{B} \cdot v\}(y)dy = \int_{\mathbb{R}^N} \eta(z) \cdot \{\boldsymbol{B} \cdot v\}(x+\varepsilon z)dz. \tag{3.3}$$

and since η is radial, and therefore $\int_{\mathbb{R}^N} \eta(z)zdz = 0$, we have

$$\frac{1}{\varepsilon}\Big(\boldsymbol{B} \cdot \{\psi_\varepsilon(x)\} - \{\boldsymbol{B} \cdot v\}(x)\Big) = \int_{\mathbb{R}^N} \eta(z)\frac{\{\boldsymbol{B} \cdot v\}(x+\varepsilon z) - \{\boldsymbol{B} \cdot v\}(x)}{\varepsilon}dz =$$
$$\int_0^1 \bigg(\int_{\mathbb{R}^N} \eta(z)z \cdot \nabla\{\boldsymbol{B} \cdot v\}(x+\varepsilon tz)dz\bigg)dt = \int_0^1 \bigg(\int_{\mathbb{R}^N} \eta(z)z \cdot \Big(\nabla\{\boldsymbol{B} \cdot v\}(x+\varepsilon tz) - \nabla\{\boldsymbol{B} \cdot v\}(x)\Big)dz\bigg)dt. \tag{3.4}$$

Then by the results of [27] and [29] we have $\lim_{\varepsilon\to0^+} \boldsymbol{A}\cdot\nabla\psi_\varepsilon = \boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N, \mathbb{R}^m)$, $\lim_{\varepsilon\to0^+} \varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\} = 0$ in $L^p(\mathbb{R}^N, \mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to0^+} \boldsymbol{B}\cdot\psi_\varepsilon = \boldsymbol{B}\cdot v$ in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^k)$ and $\lim_{\varepsilon\to0^+}(\boldsymbol{B}\cdot\psi_\varepsilon - \boldsymbol{B}\cdot v)/\varepsilon = 0$ in $L^p(\mathbb{R}^N, \mathbb{R}^k)$ for every $p \geq 1$. Moreover, $\boldsymbol{A}\cdot\nabla\psi_\varepsilon$, $\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\}$, $\boldsymbol{B}\cdot\psi_\varepsilon$ and $\nabla\{\boldsymbol{B}\cdot\psi_\varepsilon\}$ are bounded in L^∞,

$$\limsup_{\varepsilon\to0^+} \frac{1}{\varepsilon}\int_\Omega \Big|\boldsymbol{A}\cdot\nabla\psi_\varepsilon(x) - \{\boldsymbol{A}\cdot\nabla v\}(x)\Big|dx < +\infty\,, \tag{3.5}$$

and the following Theorem holds.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Furthermore, let $\boldsymbol{A} \in \mathcal{L}(\mathbb{R}^{d\times N}; \mathbb{R}^m)$, $\boldsymbol{B} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^k)$ and let $F \in C^1(\mathbb{R}^{m\times N} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^q, \mathbb{R})$, satisfying $F \geq 0$. Let $f \in BV_{loc}(\mathbb{R}^N, \mathbb{R}^q) \cap L^\infty$ and $v \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^d)$ be such that $\boldsymbol{A}\cdot\nabla v \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^m)$ and $\boldsymbol{B}\cdot v \in Lip(\mathbb{R}^N, \mathbb{R}^k) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^k) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^k)$, $\|D(\boldsymbol{A}\cdot\nabla v)\|(\partial\Omega) = 0$ and $F\big(0, \{\boldsymbol{A}\cdot\nabla v\}(x), \{\boldsymbol{B}\cdot v\}(x), f(x)\big) = 0$ a.e. in Ω. Consider ψ_ε, defined by (3.1). Then,*

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon}\int_\Omega F\Big(\varepsilon\nabla\big\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon(x)\big\},\, \boldsymbol{A}\cdot\nabla\psi_\varepsilon(x),\, \boldsymbol{B}\cdot\psi_\varepsilon(x),\, f(x)\Big)dx = \int_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}} \Bigg\{$$
$$\int_{\mathbb{R}} F\bigg(p(t,x)\Big(\{\boldsymbol{A}\cdot\nabla v\}^+(x) - \{\boldsymbol{A}\cdot\nabla v\}^-(x)\Big)\otimes\boldsymbol{\nu}(x), \Gamma(t,x), \{\boldsymbol{B}\cdot v\}(x), \frac{(|t|+t)f^-(x) + (|t|-t)f^+(x)}{2|t|}\bigg)dt$$
$$\Bigg\}d\mathcal{H}^{N-1}(x), \tag{3.6}$$

(with $\boldsymbol{\nu}(x)$ denoting the orientation vector of $J_{\boldsymbol{A}\cdot\nabla v}$) where

$$\Gamma(t,x) := \bigg(\int_{-\infty}^t p(s,x)ds\bigg)\{\boldsymbol{A}\cdot\nabla v\}^-(x) + \bigg(\int_t^{+\infty} p(s,x)ds\bigg)\{\boldsymbol{A}\cdot\nabla v\}^+(x)\,, \tag{3.7}$$

with $p(t,x)$ is defined by

$$p(t,x) := \int_{H^0_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x) + y)\,d\mathcal{H}^{N-1}(y)\,, \tag{3.8}$$

and we assume that the orientation of J_f coincides with the orientation of $J_{\boldsymbol{A}\cdot\nabla v}$ $\mathcal{H}^{N-1}$ a.e. on $J_f \cap J_{\boldsymbol{A}\cdot\nabla v}$.

3.2 Modification of the primary sequence near the single elementary surface

Next assume we are in the settings of Theorem 3.1. Let S be $(N-1)$-dimensional hypersurface satisfying $\overline{S} \subset\subset \Omega$. Moreover, assume that for some function $g(x') \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ and a bounded open set $\mathcal{U} \subset \mathbb{R}^{N-1}$ we have

$$S = \{x = (x_1, x') : \, x' \in \mathcal{U}, \; x_1 = g(x')\} , \tag{3.9}$$

and

$$\boldsymbol{n}(x') = (1, -\nabla_{x'} g(x'))/\sqrt{1 + |\nabla_{x'} g(x')|^2} , \tag{3.10}$$

where $\boldsymbol{n}(x')$ is a normal vector to S at the point $\big(g(x'), x'\big)$. Using Theorem A.1 we deduce that there exists a set $D \subset S$ such that $\mathcal{H}^{N-1}(S \setminus D) = 0$ and for every $x \in D$ we have

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho^+(x, \boldsymbol{n}(x'))} \Big(\big|\{\boldsymbol{A} \cdot \nabla v\}(y) - \{\boldsymbol{A} \cdot \nabla v\}^+(x)\big| + \big|f(y) - f^+(x)\big| \Big)\, dy}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0 ,$$
$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho^-(x, \boldsymbol{n}(x'))} \Big(\big|\{\boldsymbol{A} \cdot \nabla v\}(y) - \{\boldsymbol{A} \cdot \nabla v\}^-(x)\big| + \big|f(y) - f^-(x)\big| \Big)\, dy}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0 , \tag{3.11}$$

where we use the convention $\nabla v^+(x) = \nabla v^-(x) = \nabla \tilde{v}(x)$ and $f^+(x) = f^-(x) = \tilde{f}(x)$ at a point of approximate continuity x.

Consider a radial function $\theta_0(z') = \kappa(|z'|) \in C_c^\infty(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\operatorname{supp} \theta_0 \subset\subset B_1(0)$, $\theta_0 \geq 0$ and $\int_{\mathbb{R}^{N-1}} \theta_0(z') dz' = 1$. Then for any $\varepsilon > 0$ define the function $g_\varepsilon(x') : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$g_\varepsilon(x') := \frac{1}{\varepsilon^{N-1}} \int_{\mathbb{R}^{N-1}} \theta_0\Big(\frac{y' - x'}{\varepsilon}\Big) g(y') dy' = \int_{\mathbb{R}^{N-1}} \theta_0(z') g(x' + \varepsilon z')\, dz', \quad \forall x' \in \mathbb{R}^{N-1} . \tag{3.12}$$

Since θ_0 is radial, we obtain $\int_{\mathbb{R}^{N-1}} \theta_0(z') z'\, dz' = 0$. Therefore, since $g \in C^1$, clearly we have

$$\frac{1}{\varepsilon} \sup_{x' \in \mathcal{U}} |g_\varepsilon(x') - g(x')| \to 0 \quad \text{as } \varepsilon \to 0^+ ,$$
$$\sup_{x' \in \mathcal{U}} |\nabla g_\varepsilon(x') - \nabla g(x')| \to 0 \quad \text{as } \varepsilon \to 0^+ ,$$
$$\sup_{x' \in \mathcal{U}} |\varepsilon \nabla^2 g_\varepsilon(x')| \to 0 \quad \text{as } \varepsilon \to 0^+ . \tag{3.13}$$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N$ be a standard orthonormal base in $\mathbb{R}^N$ and let $h_0(y, x')$ be an arbitrary $C^\infty(\mathbb{R}^N \times \mathbb{R}^{N-1}, \mathbb{R}^d)$ function satisfying

$$h_0(y, x') = 0 \;\text{ if }\; |y_1| \geq 1/2, \;\text{ and }\; h_0(y + \boldsymbol{e}_j, x') = h_0(y, x') \;\; \forall j = 2, \ldots, N , \tag{3.14}$$

and

$$\overline{\operatorname{supp} h_0(y, x')} \subset \mathbb{R}^N \times \mathcal{U} . \tag{3.15}$$

Denote the set of such functions by $\mathcal{P}(\mathcal{U})$. Let $L > 0$ be an arbitrary number and let

$$h(y, x') := \frac{1}{L} h_0(Ly, x') . \tag{3.16}$$

Then $h \in C^\infty(\mathbb{R}^N \times \mathbb{R}^{N-1}, \mathbb{R}^d)$ satisfies

$$h(y, x') = 0 \;\text{ if }\; |y_1| \geq 1/(2L), \;\text{ and }\; h\big(y + (1/L)\boldsymbol{e}_j, x'\big) = h(y, x') \;\; \forall j = 2, \ldots, N , \tag{3.17}$$

and

$$\overline{\operatorname{supp} h(y, x')} \subset \mathbb{R}^N \times \mathcal{U} . \tag{3.18}$$

For any $\varepsilon > 0$ define the function $u_\varepsilon(x) \in C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ by

$$u_\varepsilon(x) := \psi_\varepsilon(x) + \varepsilon h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) , \tag{3.19}$$

where ψ_ε is defined by (3.2).

Lemma 3.1. *Let S, g, $\mathcal{U}$, $\boldsymbol{n}$, θ_0, g_ε, $\mathcal{P}(\mathcal{U})$, $h_0 \in \mathcal{P}(\mathcal{U})$, L, h and u_ε be as above. Then* $\lim_{\varepsilon\to0^+} \boldsymbol{A}\cdot\nabla u_\varepsilon = \boldsymbol{A}\cdot\nabla v$ *in* $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+} \varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\} = 0$ *in* $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to0^+} \boldsymbol{B}\cdot u_\varepsilon = \boldsymbol{B}\cdot v$ *in* $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$ *and* $\lim_{\varepsilon\to0^+}(\boldsymbol{B}\cdot u_\varepsilon - \boldsymbol{B}\cdot v)/\varepsilon = 0$ *in* $L^p(\mathbb{R}^N,\mathbb{R}^k)$ *for every $p \geq 1$. Moreover,* $\boldsymbol{A}\cdot\nabla u_\varepsilon$, $\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\}$, $\boldsymbol{B}\cdot u_\varepsilon$ *and* $\nabla\{\boldsymbol{B}\cdot u_\varepsilon\}$ *are bounded in L^∞, and since for small $\varepsilon > 0$ we have* $\int_\Omega \boldsymbol{A}\cdot\nabla u_\varepsilon dx = \int_\Omega \boldsymbol{A}\cdot\nabla\psi_\varepsilon dx$ *by (3.5), we obtain*

$$\limsup_{\varepsilon\to0^+}\left|\frac{1}{\varepsilon}\bigg(\int_\Omega \boldsymbol{A}\cdot\nabla u_\varepsilon(x)\,dx - \int_\Omega\{\boldsymbol{A}\cdot\nabla v\}(x)\,dx\bigg)\right| < +\infty\,, \tag{3.20}$$

Proof. Denote

$$r_\varepsilon(x) := \varepsilon h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg)\,. \tag{3.21}$$

It is clear that there exists $M > 0$ such that

$$\left|\frac{r_\varepsilon(x)}{\varepsilon}\right| + \Big|\nabla r_\varepsilon(x)\Big| + \Big|\varepsilon\nabla^2 r_\varepsilon(x)\Big| \leq M \quad \forall x\in\mathbb{R}^N,\ \forall\varepsilon > 0. \tag{3.22}$$

It is sufficient to prove that $\lim_{\varepsilon\to0^+}\varepsilon\nabla r_\varepsilon^2 = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{d\times N\times N})$, $\lim_{\varepsilon\to0^+}\nabla r_\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{d\times N})$ and $\lim_{\varepsilon\to0^+} r_\varepsilon/\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^d)$ for every $p \geq 1$. Indeed, by the first equality in (3.17) we have

$$r_\varepsilon(x) = \varepsilon h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) = 0 \quad\text{if}\quad \big|x_1 - g_\varepsilon(x')\big| > \frac{\varepsilon}{2L}.$$

Therefore,

$$r_\varepsilon(x) = 0,\ \ \nabla r_\varepsilon(x) = 0 \text{ and } \nabla^2 r_\varepsilon(x) = 0 \quad\text{if}\quad \big|x_1 - g_\varepsilon(x')\big| > \frac{\varepsilon}{2L}. \tag{3.23}$$

Thus, by (3.18), (3.22) and (3.23) we obtain that there exists a compact $\widetilde{K} \subset\subset \mathbb{R}^{N-1}$, independent on ε, such that

$$\int\limits_{\mathbb{R}^N}\left(\left|\frac{r_\varepsilon(x)}{\varepsilon}\right|^p + \Big|\nabla r_\varepsilon(x)\Big|^p + \Big|\varepsilon\nabla^2 r_\varepsilon(x)\Big|^p\right)dx =$$

$$\int\limits_{\widetilde{K}}\int\limits_{g_\varepsilon(x')-\frac{\varepsilon}{2L}}^{g_\varepsilon(x')+\frac{\varepsilon}{2L}}\left(\left|\frac{r_\varepsilon(x)}{\varepsilon}\right|^p + \Big|\nabla r_\varepsilon(x)\Big|^p + \Big|\varepsilon\nabla^2 r_\varepsilon(x)\Big|^p\right)dx_1dx' \leq \frac{3M^p}{L}\,\varepsilon\mathcal{L}^{(N-1)}(\widetilde{K}) \to 0 \quad\text{as}\quad \varepsilon\to0^+.$$

□

Proposition 3.1. *Let S, g, $\mathcal{U}$, $\boldsymbol{n}$, θ_0, g_ε, $\mathcal{P}(\mathcal{U})$, $h_0 \in \mathcal{P}(\mathcal{U})$, L, h and u_ε be as above and let v F, f, $\boldsymbol{A}$, $\boldsymbol{B}$, ψ_ε, p and Γ be the same as in Theorem 3.1. Then*

$$\lim_{\varepsilon\to0}\int\limits_\Omega\frac{1}{\varepsilon}\Big\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big) - F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\Big\}\,dx = \int\limits_S\int\limits_{\mathbb{R}}\int\limits_{I_1^{N-1}}\Bigg\{$$

$$\frac{1}{L}F\bigg(p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{n}(x'),\,\Gamma(-s_1/L,x),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)$$

$$-\frac{1}{L}F\bigg(p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{n}(x') + L\nabla_y\{\boldsymbol{A}\cdot\nabla_y\tilde{h}\}\big(Q_{x'}(s),x'\big),$$

$$\Gamma(-s_1/L,x) + \boldsymbol{A}\cdot\nabla_y\tilde{h}\big(Q_{x'}(s),x'\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)\Bigg\}\,ds'ds_1\,d\mathcal{H}^{N-1}(x)\,, \tag{3.24}$$

where $(s_1, s') := s \in \mathbb{R}\times\mathbb{R}^{N-1}$,

$$I_1^{N-1} := \Big\{s' = (s_2,\dots s_N)\in\mathbb{R}^{N-1} : -1/2 \leq s_j \leq 1/2 \text{ if } 2\leq j\leq N\Big\}$$

$\tilde{h}(y,x') \in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$ *is given by*

$$\tilde{h}(y,x') := h_0\Big(\{y_1 - \nabla_{x'}g(x')\cdot y', y'\}, x'\Big)\,, \tag{3.25}$$

$$\zeta(t,a,b) := \frac{(|t|+t)a + (|t|-t)b}{2|t|} = \begin{cases} a & \text{if } t > 0\,, \\ b & \text{if } t < 0\,, \end{cases} \tag{3.26}$$

and the linear transformation $Q_{x'}(s) = Q_{x'}(s_1, s_2, \ldots, s_N) : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$Q_{x'}(s) := \sum_{j=1}^{N} s_j\, \boldsymbol{q}_j(x')\,, \tag{3.27}$$

with

$$\boldsymbol{q}_j(x') := \begin{cases} \boldsymbol{n}(x') & \text{if} \quad j = 1\,, \\ \big(\nabla_x g(x') \cdot \boldsymbol{e}_j\big)\, \boldsymbol{e}_1 + \boldsymbol{e}_j & \text{if} \quad 2 \le j \le N\,. \end{cases} \,. \tag{3.28}$$

Proof. Observe that

$$\nabla_x u_\varepsilon(x) = \nabla_x \psi_\varepsilon(x) + \nabla_y h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \\ - \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') + \varepsilon \nabla_2 h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg), \tag{3.29}$$

where $\nabla_2 h(y, x') := \big(0, \nabla_{x'} h(y, x')\big)$ (the $\mathbb{R}^N$-gradient by the second variable) and

$$\varepsilon \nabla_x^2 u_\varepsilon(x) = \varepsilon \nabla_x^2 \psi_\varepsilon(x) + \nabla_y^2 h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) - \nabla_y \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') \\ - \nabla_x g_\varepsilon(x') \otimes \nabla_y \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) + \partial^2_{y_1 y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') \otimes \nabla_x g_\varepsilon(x') \\ - \varepsilon \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x^2 g_\varepsilon(x') + \varepsilon \nabla_2 \nabla_y h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \\ + \varepsilon \nabla_y \nabla_2 h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) - \varepsilon \nabla_2 \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') \\ - \varepsilon \nabla_x g_\varepsilon(x') \otimes \nabla_2 \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) + \varepsilon^2 \nabla_2^2 h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg). \tag{3.30}$$

Then by (3.19), (3.29) and (3.30), and by (3.2) and (3.3) we have

$$\boldsymbol{B} \cdot u_\varepsilon(x) = \int_{\mathbb{R}^N} \eta(z) \{\boldsymbol{B} \cdot v\}(x + \varepsilon z)\, dz + \varepsilon \boldsymbol{B} \cdot h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg), \tag{3.31}$$

$$\boldsymbol{A} \cdot \nabla_x u_\varepsilon(x) = \int_{\mathbb{R}^N} \eta(z) \{\boldsymbol{A} \cdot \nabla v\}(x + \varepsilon z)\, dz + \boldsymbol{A} \cdot \nabla_y h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \\ - \boldsymbol{A} \cdot \bigg\{\partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x')\bigg\} + \varepsilon \boldsymbol{A} \cdot \nabla_2 h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg), \tag{3.32}$$

and

$$\varepsilon \nabla_x \{\boldsymbol{A} \cdot \nabla_x u_\varepsilon\}(x) = -\int_{\mathbb{R}^N} \{\boldsymbol{A} \cdot \nabla v\}(x + \varepsilon z) \otimes \nabla_z \eta(z)\, dz + \nabla_y \{\boldsymbol{A} \cdot \nabla_y h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \\ - \{\boldsymbol{A} \cdot \nabla_y \partial_{y_1} h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') - \boldsymbol{A} \cdot \bigg\{\nabla_x g_\varepsilon(x') \otimes \nabla_y \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg)\bigg\} \\ + \boldsymbol{A} \cdot \bigg\{\partial^2_{y_1 y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x')\bigg\} \otimes \nabla_x g_\varepsilon(x') \\ - \varepsilon \boldsymbol{A} \cdot \bigg\{\partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x^2 g_\varepsilon(x')\bigg\} + \varepsilon \nabla_y \{\boldsymbol{A} \cdot \nabla_2 h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \\ + \varepsilon \nabla_2 \{\boldsymbol{A} \cdot \nabla_y h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) - \varepsilon \{\boldsymbol{A} \cdot \nabla_2 \partial_{y_1} h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg) \otimes \nabla_x g_\varepsilon(x') \\ - \varepsilon \boldsymbol{A} \cdot \bigg\{\nabla_x g_\varepsilon(x') \otimes \nabla_2 \partial_{y_1} h\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg)\bigg\} + \varepsilon^2 \nabla_2 \{\boldsymbol{A} \cdot \nabla_2 h\}\bigg(\Big(\frac{x_1 - g_\varepsilon(x')}{\varepsilon}, \frac{x'}{\varepsilon}\Big), x'\bigg), \tag{3.33}$$

where we denote

$$\boldsymbol{A}\cdot\Big\{\sigma\otimes\nabla^2_x g_\varepsilon(x')\Big\}:=\bigg\{\bigg(\boldsymbol{A}\cdot\Big\{\sigma\otimes\nabla_x\partial_{x_j}g_\varepsilon(x')\Big\}\bigg)_i\bigg\}_{1\leq i\leq m,\,1\leq j\leq N}\qquad\forall\sigma\in\mathbb{R}^d\,,$$

and

$$\boldsymbol{A}\cdot\Big\{\nabla_x g_\varepsilon(x')\otimes\nabla\varpi\Big\}:=\bigg\{\bigg(\boldsymbol{A}\cdot\Big\{\partial_j\varpi\otimes\nabla_x g_\varepsilon(x')\Big\}\bigg)_i\bigg\}_{1\leq i\leq m,\,1\leq j\leq N}\qquad\forall\varpi:\mathbb{R}^N\to\mathbb{R}^d\,.$$

Note also that

$$\big\{x\in\Omega: u_\varepsilon(x)\neq\psi_\varepsilon(x)\,\big\}\subset\{x:\ x'\in\mathcal{U}\ \ |x_1-g_\varepsilon(x')|<\varepsilon/(2L)\}\,.\tag{3.34}$$

Thus

$$\int\limits_\Omega\frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\bigg\}dx=$$
$$\int\limits_{\mathcal{U}}\int\limits_{g_\varepsilon(x')-\varepsilon/(2L)}^{g_\varepsilon(x')+\varepsilon/(2L)}\frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\bigg\}dx_1dx'\,.\tag{3.35}$$

Then changing variables gives

$$\int\limits_\Omega\frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\bigg\}dx=$$
$$\int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\Bigg\{F\bigg(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big),\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big),\,\boldsymbol{B}\cdot\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big),\,f\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)\bigg)$$
$$-F\bigg(\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big),\,\boldsymbol{A}\cdot\nabla u_\varepsilon(x+g_\varepsilon\big(x')\boldsymbol{e}_1\big),\,\boldsymbol{B}\cdot u_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big),\,f\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)\bigg)\Bigg\}dx_1dx'\,.\tag{3.36}$$

We also observe that for any small $\varepsilon>0$ we have

$$\boldsymbol{B}\cdot\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=\int_{\mathbb{R}^N}\eta(z)\{\boldsymbol{B}\cdot v\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1+\varepsilon z\big)\,dz=$$
$$\int_{\mathbb{R}^N}\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\{\boldsymbol{B}\cdot v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)\,dz\,,\tag{3.37}$$

and then

$$\boldsymbol{B}\cdot u_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=\boldsymbol{B}\cdot\psi_\varepsilon(x+g_\varepsilon(x')\boldsymbol{e}_1\big)+\varepsilon\,\boldsymbol{B}\cdot h(x/\varepsilon,x')=$$
$$\int_{\mathbb{R}^N}\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\{\boldsymbol{B}\cdot v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)\,dz+\varepsilon\,\boldsymbol{B}\cdot h(x/\varepsilon,x')\,.\tag{3.38}$$

Moreover,

$$\boldsymbol{A}\cdot\nabla\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=\int_{\mathbb{R}^N}\eta(z)\{\boldsymbol{A}\cdot\nabla v\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1+\varepsilon z\big)\,dz$$
$$=\int_{\mathbb{R}^N}\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\{\boldsymbol{A}\cdot\nabla v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)\,dz\,,\tag{3.39}$$

and then by (3.32) we infer

$$\boldsymbol{A}\cdot\nabla u_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=\int_{\mathbb{R}^N}\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\{\boldsymbol{A}\cdot\nabla v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)\,dz$$
$$+\boldsymbol{A}\cdot\nabla_y h(x/\varepsilon,x')-\boldsymbol{A}\cdot\Big\{\partial_{y_1}h(x/\varepsilon,x')\otimes\nabla_x g_\varepsilon(x')\Big\}+\varepsilon\boldsymbol{A}\cdot\nabla_2 h(x/\varepsilon,x')\,.\tag{3.40}$$

Finally

$$\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=-\int_{\mathbb{R}^N}\{\boldsymbol{A}\cdot\nabla v\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1+\varepsilon z\big)\otimes\nabla\eta(z)\,dz=$$
$$-\int_{\mathbb{R}^N}\{\boldsymbol{A}\cdot\nabla v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)\otimes\nabla\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\,dz\,,\quad(3.41)$$

and then by (3.33) we have

$$\begin{aligned}\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)=&\\ -\int_{\mathbb{R}^N}\{\boldsymbol{A}\cdot\nabla v\}\big(g(x')+\varepsilon z_1,x'+\varepsilon z'\big)&\otimes\nabla\eta\Big(z_1-x_1/\varepsilon+\big(g(x')-g_\varepsilon(x')\big)/\varepsilon,z'\Big)\,dz\\ +\nabla_y\{\boldsymbol{A}\cdot\nabla_y h\}(x/\varepsilon,x')-\{\boldsymbol{A}\cdot\nabla_y\partial_{y_1}h\}(x/\varepsilon,x')&\otimes\nabla_x g_\varepsilon(x')-\boldsymbol{A}\cdot\Big\{\nabla_x g_\varepsilon(x')\otimes\nabla_y\partial_{y_1}h(x/\varepsilon,x')\Big\}\\ +\boldsymbol{A}\cdot\Big\{\partial^2_{y_1y_1}h(x/\varepsilon,x')\otimes\nabla_x g_\varepsilon(x')\Big\}&\otimes\nabla_x g_\varepsilon(x')-\boldsymbol{A}\cdot\Big\{\partial_{y_1}h(x/\varepsilon,x')\otimes\{\varepsilon\nabla^2_x g_\varepsilon(x')\}\Big\}\\ +\varepsilon\nabla_y\{\boldsymbol{A}\cdot\nabla_2 h\}(x/\varepsilon,x')+\varepsilon\nabla_2\{\boldsymbol{A}\cdot\nabla_y h\}&(x/\varepsilon,x')-\varepsilon\{\boldsymbol{A}\cdot\nabla_2\partial_{y_1}h\}(x/\varepsilon,x')\otimes\nabla_x g_\varepsilon(x')\\ -\varepsilon\boldsymbol{A}\cdot\Big\{\nabla_x g_\varepsilon(x')\otimes\nabla_2\partial_{y_1}&h(x/\varepsilon,x')\Big\}+\varepsilon^2\nabla_2\{\boldsymbol{A}\cdot\nabla_2 h\}(x/\varepsilon,x')\,.\end{aligned}\quad(3.42)$$

Set

$$\delta(x_1,x'):=\Bigg(\int\limits_{H_+\big((g(x'),x'),\boldsymbol{n}(x')\big)}\eta\big(z_1-x_1,z'\big)dz\Bigg)\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big)$$
$$+\Bigg(\int\limits_{H_-\big((g(x'),x'),\boldsymbol{n}(x')\big)}\eta\big(z_1-x_1,z'\big)dz\Bigg)\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)\,,\quad(3.43)$$

and

$$\theta(x_1,x'):=-\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big)\otimes\Bigg(\int\limits_{H_+\big((g(x'),x'),\boldsymbol{n}(x')\big)}\nabla\eta\big(z_1-x_1,z'\big)dz\Bigg)$$
$$-\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)\otimes\Bigg(\int\limits_{H_-\big((g(x'),x'),\boldsymbol{n}(x')\big)}\nabla_z\eta\big(z_1-x_1,z'\big)dz\Bigg)\,,\quad(3.44)$$

where $H_+(x,\boldsymbol{n})=\{y\in\mathbb{R}^N:(y-x)\cdot\boldsymbol{n}>0\}$ and $H_-(x,\boldsymbol{n})=\{y\in\mathbb{R}^N:(y-x)\cdot\boldsymbol{n}<0\}$, and define

$$\Lambda_\varepsilon(x):=\delta(x_1/\varepsilon,x')+\boldsymbol{A}\cdot\nabla_y h(x/\varepsilon,x')-\boldsymbol{A}\cdot\Big\{\partial_{y_1}h(x/\varepsilon,x')\otimes\nabla_x g(x')\Big\}\,,\quad(3.45)$$

and

$$\Theta_\varepsilon(x):=\theta(x_1/\varepsilon,x')+\nabla_y\{\boldsymbol{A}\cdot\nabla_y h(x/\varepsilon,x')\}-\{\boldsymbol{A}\cdot\nabla_y\partial_{y_1}h\}(x/\varepsilon,x')\otimes\nabla_x g(x')$$
$$-\boldsymbol{A}\cdot\Big\{\nabla_x g(x')\otimes\nabla_y\partial_{y_1}h(x/\varepsilon,x')\Big\}+\boldsymbol{A}\cdot\Big\{\partial^2_{y_1y_1}h(x/\varepsilon,x')\otimes\nabla_x g(x')\Big\}\otimes\nabla_x g(x')\,.\quad(3.46)$$

Then by the fact that $\{\boldsymbol{B}\cdot v\}\in Lip$ and by (3.11), (3.13), (3.37), (3.39) and (3.41) we deduce

$$\int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\bigg\{\Big|\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\theta(x_1/\varepsilon,x')\Big|+\Big|\boldsymbol{A}\cdot\nabla\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\delta(x_1/\varepsilon,x')\Big|$$
$$+\Big|\boldsymbol{B}\cdot\psi_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big)\Big|\bigg\}\,dx_1dx'\to 0\quad\text{as }\varepsilon\to 0\,,\quad(3.47)$$

and by the fact that $\{\boldsymbol{B}\cdot v\}\in Lip$ and by (3.11), (3.13), (3.38), (3.40) and (3.42) we deduce

$$\int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\bigg\{\Big|\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\}\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\Theta_\varepsilon(x)\Big|+\Big|\boldsymbol{A}\cdot\nabla u_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\Lambda_\varepsilon(x)\Big| \\ +\Big|\boldsymbol{B}\cdot u_\varepsilon\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big)\Big|\bigg\}dx_1dx'\to 0\quad\text{as }\varepsilon\to 0\,, \tag{3.48}$$

Therefore, by (3.36), (3.47) and (3.48) we infer

$$\int\limits_\Omega\frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\bigg\}dx= \\ \int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\bigg\{F\Big(\theta(x_1/\varepsilon,x'),\,\delta(x_1/\varepsilon,x'),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,f\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)\Big) \\ -F\Big(\Theta_\varepsilon(x),\,\Lambda_\varepsilon(x),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,f\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)\Big)\bigg\}dx_1dx'+l_\varepsilon\,, \tag{3.49}$$

where $\lim_{\varepsilon\to0}l_\varepsilon=0$. Next by Theorem 3.108 and Remark 3.109 from [4] we deduce that

$$\lim_{\rho\to0^+}\frac{1}{\rho}\int_{-\rho}^{\rho}\Big|f\big(g(x')+s,x'\big)-\zeta\Big(s,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\Big|ds=0\qquad\text{for }\mathcal{L}^{N-1}\text{ a.e. }x'\in\mathcal{U}\,, \tag{3.50}$$

where $\zeta(t,a,b)$ is defined by (3.26). Then, since $f\in L^\infty$, by (3.50) and (3.13) we obtain

$$\int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\Big|f\big(x+g_\varepsilon(x')\boldsymbol{e}_1\big)-\zeta\Big(x_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\Big|dx_1dx'\to0\quad\text{as }\varepsilon\to0\,. \tag{3.51}$$

Thus, using (3.51), by (3.49) we obtain

$$\int\limits_\Omega\frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\bigg\}dx= \\ \int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\Bigg\{F\bigg(\theta(x_1/\varepsilon,x'),\,\delta(x_1/\varepsilon,x'),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,\zeta\Big(x_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg) \\ -F\bigg(\Theta_\varepsilon(x),\,\Lambda_\varepsilon(x),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,\zeta\Big(x_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)\Bigg\}dx_1dx'+\bar{l}_\varepsilon= \\ \int\limits_{\mathcal{K}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)}\frac{1}{\varepsilon}\Bigg\{F\bigg(\theta(x_1/\varepsilon,x'),\,\delta(x_1/\varepsilon,x'),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,\zeta\Big(x_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg) \\ -F\bigg(\Theta_\varepsilon(x),\,\Lambda_\varepsilon(x),\,\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\,\zeta\Big(x_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)\Bigg\}dx_1dx'+\bar{l}_\varepsilon\,, \tag{3.52}$$

where $\lim_{\varepsilon\to0}\bar{l}_\varepsilon=0$ and $\mathcal{K}\subset\subset\mathcal{U}$ is a compact set, such that

$$\operatorname{supp}h(y,x')\subset\mathbb{R}^N\times\mathcal{K}\,.$$

Next for every $y'\in\mathbb{R}^{N-1}$ set

$$\bar{\Lambda}_\varepsilon(x,y'):=\delta(x_1/\varepsilon,x')+\boldsymbol{A}\cdot\nabla_1h\Big(\{x_1/\varepsilon,x'/\varepsilon+y'\},x'\Big)-\boldsymbol{A}\cdot\Big\{\partial_{y_1}h\Big(\{x_1/\varepsilon,x'/\varepsilon+y'\},x'\Big)\otimes\nabla_xg(x')\Big\}\,, \tag{3.53}$$

and

$$
\bar{\Theta}_\varepsilon(x,y') := \theta(x_1/\varepsilon, x') + \nabla_1\{\boldsymbol{A}\cdot\nabla_1 h\}\Big(\{x_1/\varepsilon, x'/\varepsilon + y'\}, x'\Big) - \{\boldsymbol{A}\cdot\nabla_1\partial_{y_1} h\}\Big(\{x_1/\varepsilon, x'/\varepsilon + y'\}, x'\Big)\otimes\nabla_x g(x')
$$
$$
-\boldsymbol{A}\cdot\Big\{\nabla_x g(x')\otimes\nabla_1\partial_{y_1} h\Big(\{x_1/\varepsilon, x'/\varepsilon+y'\}, x'\Big)\Big\} + \boldsymbol{A}\cdot\Big\{\partial^2_{y_1y_1} h\Big(\{x_1/\varepsilon, x'/\varepsilon+y'\}, x'\Big)\otimes\nabla_x g(x')\Big\}\otimes\nabla_x g(x'), \tag{3.54}
$$

where we denote $\nabla_1 h(y,x') := \nabla_y h(y,x')$ (the partial gradient of h by the first variable). Then $\bar{\Lambda}_\varepsilon(x,0) = \Lambda_\varepsilon(x)$ and $\bar{\Theta}_\varepsilon(x,0) = \Theta_\varepsilon(x)$. Moreover, for every $y'\in\mathbb{R}^{N-1}$ and for $\varepsilon > 0$ sufficiently small, changing variables in the integral gives

$$
\int\limits_{\mathcal{K}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} F\bigg(\bar{\Theta}_\varepsilon(x,y'),\, \bar{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\,dx_1dx' =
$$
$$
\int\limits_{\mathcal{K}+\varepsilon y'}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} F\bigg(\tilde{\Theta}_\varepsilon(x,y'),\, \tilde{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big),\, \zeta\Big(x_1, f^+\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big)\Big)\bigg)\,dx_1dx'
$$
$$
= \int\limits_{\mathcal{K}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} F\bigg(\tilde{\Theta}_\varepsilon(x,y'),\, \tilde{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big),\, \zeta\Big(x_1, f^+\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big)\Big)\bigg)\,dx_1dx' + \varepsilon\tilde{l}^{(0)}_\varepsilon(y'), \tag{3.55}
$$

where,

$$
\tilde{\Lambda}_\varepsilon(x,y') := \delta(x_1/\varepsilon, x'-\varepsilon y') + \boldsymbol{A}\cdot\nabla_1 h(x/\varepsilon, x'-\varepsilon y') - \boldsymbol{A}\cdot\Big\{\partial_{y_1} h(x/\varepsilon, x'-\varepsilon y')\otimes\nabla_x g(x'-\varepsilon y')\Big\}, \tag{3.56}
$$

$$
\tilde{\Theta}_\varepsilon(x,y') := \theta(x_1/\varepsilon, x'-\varepsilon y') + \nabla_1\{\boldsymbol{A}\cdot\nabla_1 h\}(x/\varepsilon, x'-\varepsilon y') - \{\boldsymbol{A}\cdot\nabla_1\partial_{y_1} h\}(x/\varepsilon, x'-\varepsilon y')\otimes\nabla_x g(x'-\varepsilon y')
$$
$$
-\boldsymbol{A}\cdot\Big\{\nabla_x g(x'-\varepsilon y')\otimes\nabla_1\partial_{y_1} h(x/\varepsilon, x'-\varepsilon y')\Big\} + \boldsymbol{A}\cdot\Big\{\partial^2_{y_1y_1} h(x/\varepsilon, x'-\varepsilon y')\otimes\nabla_x g(x'-\varepsilon y')\Big\}\otimes\nabla_x g(x'-\varepsilon y'), \tag{3.57}
$$

and $\tilde{l}^{(0)}_\varepsilon(y') \to 0$ as $\varepsilon\to 0^+$ and $\|\tilde{l}^{(0)}_\varepsilon(y')\|_{L^\infty(\mathcal{D})} < C$ for every bounded set $\mathcal{D}\subset\subset\mathbb{R}^{N-1}$. On the other hand, since $\delta,\theta\in L^1_{loc}$, we deduce that

$$
\int\limits_{\mathcal{U}}\int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \frac{1}{\varepsilon}\Big(\big|\theta(x_1/\varepsilon, x'-\varepsilon y') - \theta(x_1/\varepsilon, x')\big| + \big|\delta(x_1/\varepsilon, x'-\varepsilon y') - \delta(x_1/\varepsilon, x')\big|\Big)dx_1dx' =
$$
$$
\int\limits_{\mathcal{U}}\int\limits_{-1/(2L)}^{1/(2L)} \Big(\big|\theta(x_1, x'-\varepsilon y') - \theta(x_1, x')\big| + \big|\delta(x_1, x'-\varepsilon y') - \delta(x_1, x')\big|\Big)dx_1dx' \to 0, \tag{3.58}
$$

as $\varepsilon\to 0^+$. Moreover, since $\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big), f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big), \nabla g(x')\in L^1_{loc}(\mathbb{R}^{N-1})$ we also have

$$
\int\limits_{\mathcal{U}}\bigg(\Big|\{\boldsymbol{B}\cdot v\}\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big) - \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big)\Big| + \Big|f^+\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big) - f^+\big(g(x'),x'\big)\Big|
$$
$$
+\Big|f^-\big(g(x'-\varepsilon y'),x'-\varepsilon y'\big) - f^-\big(g(x'),x'\big)\Big| + \Big|\nabla g(x'-\varepsilon y') - \nabla g(x')\Big|\bigg)dx' \to 0, \tag{3.59}
$$

as $\varepsilon \to 0^+$. Therefore, by (3.58), (3.59) and (3.55) we deduce

$$\Bigg| \int\limits_{\mathcal{K}} \int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \frac{1}{\varepsilon} F\bigg(\bar{\Theta}_\varepsilon(x,y'),\, \bar{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\,dx_1dx'$$
$$- \int\limits_{\mathcal{K}} \int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \frac{1}{\varepsilon} F\bigg(\Theta_\varepsilon(x),\, \Lambda_\varepsilon(x),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\,dx_1dx' \Bigg| = \tilde{l}_\varepsilon(y')\,, \tag{3.60}$$

where $\tilde{l}_\varepsilon(y') \to 0$ as $\varepsilon \to 0^+$ and $\|\tilde{l}_\varepsilon(y')\|_{L^\infty(\mathcal{D})} < C$ for every bounded set $\mathcal{D} \subset\subset \mathbb{R}^{N-1}$. Then by (3.52) and (3.60) we infer

$$\int\limits_\Omega \frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\, \boldsymbol{A}\cdot\nabla\psi_\varepsilon,\, \boldsymbol{B}\cdot\psi_\varepsilon,\, f\Big) - F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\, \boldsymbol{A}\cdot\nabla u_\varepsilon,\, \boldsymbol{B}\cdot u_\varepsilon,\, f\Big)\bigg\}\,dx =$$
$$\int\limits_{\mathcal{K}} \int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \int_{I_L^{N-1}} \frac{L^{N-1}}{\varepsilon}\Bigg\{F\bigg(\theta(x_1/\varepsilon,x'),\, \delta(x_1/\varepsilon,x'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)$$
$$- F\bigg(\bar{\Theta}_\varepsilon(x,y'),\, \bar{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\Bigg\}\,dy'dx_1dx' + \tilde{l}_\varepsilon =$$
$$\int\limits_{\mathcal{U}} \int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \int_{I_L^{N-1}} \frac{L^{N-1}}{\varepsilon}\Bigg\{F\bigg(\theta(x_1/\varepsilon,x'),\, \delta(x_1/\varepsilon,x'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)$$
$$- F\bigg(\bar{\Theta}_\varepsilon(x,y'),\, \bar{\Lambda}_\varepsilon(x,y'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\Bigg\}\,dy'dx_1dx' + \tilde{l}_\varepsilon\,, \tag{3.61}$$

where $\lim_{\varepsilon\to 0}\tilde{l}_\varepsilon = 0$ and $I_L^{N-1} := \{y' \in \mathbb{R}^{N-1} : -1/(2L) \le y'_j \le 1/(2L) \text{ if } 1 \le j \le (N-1)\}$. Next since for every locally integrable function $P : \mathbb{R}^{N-1} \to \mathbb{R}$ satisfying $P\big(y'_1, y'_2, \dots (y'_j + 1/L), \dots y'_{N-1}\big) = P\big(y'_1, y'_2, \dots y'_j, \dots y'_{N-1}\big)$ for every $1 \le j \le N-1$ we have

$$\int_{I_L^{N-1}} P(y'+z')dy' = \int_{I_L^{N-1}} P(y')dy' \quad \forall z' \in \mathbb{R}^{N-1}\,,$$

by (3.17) and (3.61) we deduce

$$\int\limits_\Omega \frac{1}{\varepsilon}\bigg\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\, \boldsymbol{A}\cdot\nabla\psi_\varepsilon,\, \boldsymbol{B}\cdot\psi_\varepsilon,\, f\Big) - F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\, \boldsymbol{A}\cdot\nabla u_\varepsilon,\, \boldsymbol{B}\cdot u_\varepsilon,\, f\Big)\bigg\}\,dx =$$
$$\int\limits_{\mathcal{U}} \int\limits_{-\varepsilon/(2L)}^{\varepsilon/(2L)} \int_{I_L^{N-1}} \frac{L^{N-1}}{\varepsilon}\Bigg\{F\bigg(\theta(x_1/\varepsilon,x'),\, \delta(x_1/\varepsilon,x'),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)$$
$$-F\bigg(\bar{\Theta}_\varepsilon(x,y'-x'/\varepsilon),\, \bar{\Lambda}_\varepsilon(x,y'-x'/\varepsilon),\, \{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\, \zeta\Big(x_1, f^+\big(g(x'),x'\big), f^-\big(g(x'),x'\big)\Big)\bigg)\Bigg\}\,dy'dx_1dx' + \tilde{l}_\varepsilon\,. \tag{3.62}$$

Therefore, changing the variables $z_1 := Lx_1/\varepsilon$, $z' := Ly'$ in (3.62) together with (3.16) gives

$$\lim_{\varepsilon\to0}\int\limits_{\Omega}\frac{1}{\varepsilon}\Big\{F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)-F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla u_\varepsilon\},\,\boldsymbol{A}\cdot\nabla u_\varepsilon,\,\boldsymbol{B}\cdot u_\varepsilon,\,f\Big)\Big\}\,dx=$$

$$\int\limits_{\mathcal{U}}\int\limits_{-1/2}^{1/2}\int_{I_1^{N-1}}\frac{1}{L}\bigg\{F\bigg(\theta(z_1/L,x'),\delta(z_1/L,x'),\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\zeta\Big(z_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)-$$

$$F\bigg(\theta(z_1/L,x')+\sigma(z,x'),\delta(z_1/L,x')+\kappa(z,x'),\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\zeta\Big(z_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)\bigg\}dz'dz_1dx'$$

$$=\int\limits_{\mathcal{U}}\int\limits_{\mathbb{R}}\int_{I_1^{N-1}}\frac{1}{L}\bigg\{F\bigg(\theta(z_1/L,x'),\delta(z_1/L,x'),\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\zeta\Big(z_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)-$$

$$F\bigg(\theta(z_1/L,x')+\sigma(z,x'),\delta(z_1/L,x')+\kappa(z,x'),\{\boldsymbol{B}\cdot v\}\big(g(x'),x'\big),\zeta\Big(z_1,f^+\big(g(x'),x'\big),f^-\big(g(x'),x'\big)\Big)\bigg)\bigg\}dz'dz_1dx',\tag{3.63}$$

where

$$\kappa(z,x'):=\boldsymbol{A}\cdot\nabla_z h_0(z,x')-\boldsymbol{A}\cdot\Big\{\partial_{z_1}h_0(z,x')\otimes\nabla_x g(x')\Big\},\tag{3.64}$$

and

$$\sigma(z,x'):=L\bigg(\nabla_z\{\boldsymbol{A}\cdot\nabla_z h_0\}(z,x')-\{\boldsymbol{A}\cdot\nabla_z\partial_{z_1}h_0\}(z,x')\otimes\nabla_x g(x')$$
$$-\boldsymbol{A}\cdot\Big\{\nabla_x g(x')\otimes\nabla_z\partial_{z_1}h_0(z,x')\Big\}+\boldsymbol{A}\cdot\Big\{\partial^2_{z_1z_1}h_0(z,x')\otimes\nabla_x g(x')\Big\}\otimes\nabla_x g(x')\bigg).\tag{3.65}$$

On the other hand we have

$$\delta(t,x')=\bigg(\int\limits_{H_+\big((g(x'),x'),\boldsymbol{n}(x')\big)}\eta\Big(z-\{t\boldsymbol{n}_1(x')\}\boldsymbol{n}(x')\Big)dz\bigg)\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big)+$$
$$\bigg(\int\limits_{H_-\big((g(x'),x'),\boldsymbol{n}(x')\big)}\eta\Big(z-\{t\boldsymbol{n}_1(x')\}\boldsymbol{n}(x')\Big)dz\bigg)\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)=\Gamma\Big(-t\boldsymbol{n}_1(x'),\{g(x'),x'\}\Big),\tag{3.66}$$

and

$$\theta(t,x')=-\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big)\otimes\bigg(\int\limits_{H_+\big((g(x'),x'),\boldsymbol{n}(x')\big)}\nabla\eta\Big(z-\{t\boldsymbol{n}_1(x')\}\boldsymbol{n}(x')\Big)dz\bigg)$$
$$-\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)\otimes\bigg(\int\limits_{H_-\big((g(x'),x'),\boldsymbol{n}(x')\big)}\nabla\eta\Big(z-\{t\boldsymbol{n}_1(x')\}\boldsymbol{n}(x')\Big)dz\bigg)$$
$$=p\Big(-t\boldsymbol{n}_1(x'),\{g(x'),x'\}\Big)\Big(\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big)-\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)\Big)\otimes\boldsymbol{n}(x'),\tag{3.67}$$

where as in (3.8) and (3.7),

$$p(t,x):=\int_{H^0_{\boldsymbol{n}(x')}}\eta(t\boldsymbol{n}(x')+y)\,d\mathcal{H}^{N-1}(y),\tag{3.68}$$

$$\Gamma(t,x):=\bigg(\int_{-\infty}^t p(s,x)ds\bigg)\cdot\{\boldsymbol{A}\cdot\nabla v\}^-\big(g(x'),x'\big)+\bigg(\int_t^{\infty}p(s,x)ds\bigg)\cdot\{\boldsymbol{A}\cdot\nabla v\}^+\big(g(x'),x'\big),\tag{3.69}$$

and by $\boldsymbol{n}_1$ we denote the first coordinate of $\boldsymbol{n}$.

Next define $\tilde h(y,x')\in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$ by

$$\tilde h(y,x'):=h_0\Big(\{y_1-\nabla_{x'}g(x')\cdot y',y'\},x'\Big),\tag{3.70}$$

where $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}$, and set

$$\boldsymbol{q}_j(x') := \begin{cases} \boldsymbol{n}(x') & \text{if} \quad j = 1\,, \\ \big(\nabla_x g(x') \cdot \boldsymbol{e}_j\big)\, \boldsymbol{e}_1 + \boldsymbol{e}_j & \text{if} \quad 2 \le j \le N\,. \end{cases} \tag{3.71}$$

Then using (3.10) by definition we have

$$\boldsymbol{q}_j(x') \cdot \boldsymbol{q}_1(x') = \boldsymbol{q}_j(x') \cdot \boldsymbol{n}(x') = 0 \qquad \text{for every} \quad 2 \le j \le N\,. \tag{3.72}$$

Moreover, by (3.14) we have

$$\tilde{h}(y, x') = 0 \quad \text{if} \quad \big|\big(y \cdot \boldsymbol{q}_1(x')\big)\big| \ge \frac{1}{2}\boldsymbol{n}_1(x'), \quad \text{and} \quad \tilde{h}\big(y + \boldsymbol{q}_j(x'), x'\big) = \tilde{h}(y, x') \quad \forall j = 2, \dots, N\,, \tag{3.73}$$

and by (3.15),

$$\operatorname{supp} \tilde{h}(y, x') \subset\subset \mathbb{R}^N \times \mathcal{U}\,. \tag{3.74}$$

Furthermore, by the definitions (3.70), (3.64), and (3.65) we deduce

$$\begin{aligned} \boldsymbol{A} \cdot \nabla_y \tilde{h}(y, x') &= \kappa\Big(\big\{y_1 - \nabla_{x'} g(x') \cdot y', y'\big\}, x'\Big)\,, \\ L\nabla_y\{\boldsymbol{A} \cdot \nabla_y \tilde{h}\}(y, x') &= \sigma\Big(\big\{y_1 - \nabla_{x'} g(x') \cdot y', y'\big\}, x'\Big)\,. \end{aligned} \tag{3.75}$$

Then, since by (3.63) we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon}\Big\{F\Big(\varepsilon\nabla\{\boldsymbol{A} \cdot \nabla\psi_\varepsilon\},\, \boldsymbol{A} \cdot \nabla\psi_\varepsilon,\, \boldsymbol{B} \cdot \psi_\varepsilon,\, f\Big) - F\Big(\varepsilon\nabla\{\boldsymbol{A} \cdot \nabla u_\varepsilon\},\, \boldsymbol{A} \cdot \nabla u_\varepsilon,\, \boldsymbol{B} \cdot u_\varepsilon,\, f\Big)\Big\}\, dx = \\ &\quad \int_{\mathcal{U}} \int_{\mathbb{R}} \int_{I_1^{N-1}} \frac{1}{L}\bigg\{F\bigg(\theta(z_1/L, x'), \delta(z_1/L, x'), \{\boldsymbol{B} \cdot v\}\big(g(x'), x'\big), \zeta\Big(z_1, f^+\big(g(x'), x'\big), f^-\big(g(x'), x'\big)\Big)\bigg) \\ &\quad - F\bigg(\theta(z_1/L, x') + \sigma(z, x'), \delta(z_1/L, x') + \kappa(z, x'), \{\boldsymbol{B} \cdot v\}\big(g(x'), x'\big), \zeta\Big(z_1, f^+\big(g(x'), x'\big), f^-\big(g(x'), x'\big)\Big)\bigg) \\ &\quad \bigg\}\, dz'\, dz_1\, dx'\,, \end{aligned} \tag{3.76}$$

changing variables $(z_1, z') = \big(y_1 - \nabla_{x'} g(x') \cdot y', y'\big)$ of the internal integration in all places in the r.h.s. of (3.76) together with (3.64), (3.65), (3.66), (3.67) and (3.75) gives

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon}\Big\{F\Big(\varepsilon\nabla\{\boldsymbol{A} \cdot \nabla\psi_\varepsilon\},\, \boldsymbol{A} \cdot \nabla\psi_\varepsilon,\, \boldsymbol{B} \cdot \psi_\varepsilon,\, f\Big) - F\Big(\varepsilon\nabla\{\boldsymbol{A} \cdot \nabla u_\varepsilon\},\, \boldsymbol{A} \cdot \nabla u_\varepsilon,\, \boldsymbol{B} \cdot u_\varepsilon,\, f\Big)\Big\}\, dx = \\ &\int_S \int_{\{y \in \mathbb{R}^N :\, y' \in I_1^{N-1}\}} \frac{1}{L\sqrt{\big(1 + |\nabla_{x'} g(x')|^2\big)}}\bigg\{F\bigg(p\big(-\boldsymbol{n}(x') \cdot y\,/L, x\big)\big(\{\boldsymbol{A} \cdot \nabla v\}^+(x) - \{\boldsymbol{A} \cdot \nabla v\}^-(x)\big) \otimes \boldsymbol{n}(x'), \\ &\qquad \Gamma\big(-\boldsymbol{n}(x') \cdot y\,/L, x\big),\, \{\boldsymbol{B} \cdot v\}(x),\, \zeta\big(\boldsymbol{n}(x') \cdot y, f^+(x), f^-(x)\big)\bigg) \\ &\qquad - F\bigg(p\big(-\boldsymbol{n}(x') \cdot y\,/L, x\big)\big(\{\boldsymbol{A} \cdot \nabla v\}^+(x) - \{\boldsymbol{A} \cdot \nabla v\}^-(x)\big) \otimes \boldsymbol{n}(x') + L\nabla_y\{\boldsymbol{A} \cdot \nabla_y \tilde{h}\}(y, x'), \\ &\qquad \Gamma\big(-\boldsymbol{n}(x') \cdot y\,/L, x\big) + \boldsymbol{A} \cdot \nabla_y \tilde{h}(y, x'),\, \{\boldsymbol{B} \cdot v\}(x),\, \zeta\big(\boldsymbol{n}(x') \cdot y, f^+(x), f^-(x)\big)\bigg)\bigg\}\, dy\, d\mathcal{H}^{N-1}(x)\,, \end{aligned} \tag{3.77}$$

Consider the linear transformation $Q_{x'}(s) = Q_{x'}(s_1, s_2, \dots, s_N) : \mathbb{R}^N \to \mathbb{R}^N$ by

$$Q_{x'}(s) := \sum_{j=1}^N s_j\, \boldsymbol{q}_j(x')\,, \tag{3.78}$$

where $\boldsymbol{q}_j$ is defined by (3.71). Then by (3.71)

$$\det\{Q_{x'}\} = \boldsymbol{n}_1(x')\Big(1+\big|\nabla_{x'}g(x')\big|^2\Big) = \sqrt{\Big(1+\big|\nabla_{x'}g(x')\big|^2\Big)}\,. \tag{3.79}$$

Moreover, we have

$$\begin{aligned}
Q_{x'}\bigg(\Big\{s\in\mathbb{R}^N:\ s_1>0,\ s_j\in\big(-1/2+\alpha_{x'}s_1,1/2+\alpha_{x'}s_1\big)\ \forall j\geq 2\Big\}\bigg)&\\
=\{\boldsymbol{n}(x')\cdot y>0\}\cap\{y'\in I_1^{N-1}\}\,,&\\
Q_{x'}\bigg(\Big\{s\in\mathbb{R}^N:\ s_1<0,\ s_j\in\big(-1/2+\alpha_{x'}s_1,1/2+\alpha_{x'}s_1\big)\ \forall j\geq 2\Big\}\bigg)&\\
=\{\boldsymbol{n}(x')\cdot y<0\}\cap\{y'\in I_1^{N-1}\}\,,&
\end{aligned} \tag{3.80}$$

where $\alpha_{x'}:=\boldsymbol{n}_1(x')\big(\nabla_x g(x')\cdot\boldsymbol{e}_j\big)$ and by (3.73) we deduce

$$\tilde{h}\big(Q_{x'}(s+\boldsymbol{e}_j),x'\big)=\tilde{h}\big(Q_{x'}(s),x'\big)\ \ \forall j=2,\dots,N\,. \tag{3.81}$$

Therefore, changing variables from y to s in (3.77) and using (3.72), (3.79), (3.80) and a periodicity condition (3.81) we deduce (3.24). □

Lemma 3.2. *Let S, g, $\mathcal{U}$, $\boldsymbol{n}$, θ_0, g_ε, $\mathcal{P}(\mathcal{U})$, h, u_ε, v F, f, $\boldsymbol{A}$, $\boldsymbol{B}$, ψ_ε, p and Γ be the same as in Proposition 3.1 and let $L>0$. Then*

$$\inf_{h_0\in\mathcal{P}(\mathcal{U})}\bigg\{\int\limits_S P_x\big(\tilde{h}(\cdot,x')\big)\,d\mathcal{H}^{N-1}(x)\bigg\}=\int\limits_S\Big\{\inf_{\sigma\in\mathcal{R}(x')}P_x\big(\sigma(\cdot)\big)\Big\}\,d\mathcal{H}^{N-1}(x)\,, \tag{3.82}$$

where

$$\begin{aligned}
P_x(\sigma):=\int\limits_{\mathbb{R}}\int\limits_{I_1^{N-1}}\frac{1}{L}F\bigg(&p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{n}(x')+L\nabla_y\{\boldsymbol{A}\cdot\nabla_y\sigma\}\big(Q_{x'}(s)\big),\\
&\Gamma(-s_1/L,x)+\boldsymbol{A}\cdot\nabla_y\sigma\big(Q_{x'}(s)\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)\,ds'ds_1\,,
\end{aligned} \tag{3.83}$$

with $(s_1,s'):=s\in\mathbb{R}\times\mathbb{R}^{N-1}$, $\tilde{h}(y,x')\in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$ is given by

$$\tilde{h}(y,x'):=h_0\Big(\big\{y_1-\nabla_{x'}g(x')\cdot y',y'\big\},x'\Big)\,, \tag{3.84}$$

$$\begin{aligned}
\mathcal{R}(x'):=\bigg\{&\sigma(y)\in C^\infty(\mathbb{R}^N,\mathbb{R}^d):\\
&\sigma(y)=0\ \ \text{if}\ \ \big|\big(y\cdot\boldsymbol{q}_1(x')\big)\big|\geq\frac{1}{2}\boldsymbol{n}_1(x'),\ \ \text{and}\ \ \sigma\big(y+\boldsymbol{q}_j(x')\big)=\sigma(y)\ \ \forall j=2,\dots,N\,,\bigg\}
\end{aligned} \tag{3.85}$$

and the linear transformation $Q_{x'}(s)=Q_{x'}(s_1,s_2,\dots,s_N):\mathbb{R}^N\to\mathbb{R}^N$ is defined by

$$Q_{x'}(s):=\sum_{j=1}^N s_j\,\boldsymbol{q}_j(x')\,, \tag{3.86}$$

with

$$\boldsymbol{q}_j(x'):=\begin{cases}\boldsymbol{n}(x') & \text{if}\quad j=1\,,\\ \big(\nabla_x g(x')\cdot\boldsymbol{e}_j\big)\,\boldsymbol{e}_1+\boldsymbol{e}_j & \text{if}\quad 2\leq j\leq N\,.\end{cases}\,. \tag{3.87}$$

Proof. First of all by (3.73), for every $h_0 \in \mathcal{P}(\mathcal{U})$ we have

$$\tilde{h}(y,x') \;=\; 0 \quad \text{if} \quad \big|\big(y\,\cdot\boldsymbol{q}_1(x')\big)\big| \;\geq\; \frac{1}{2}\boldsymbol{n}_1(x'), \quad \text{and} \quad \tilde{h}\big(y+\boldsymbol{q}_j(x'),x'\big) \;=\; \tilde{h}(y,x') \quad \forall j \;=\; 2,\ldots,N\,. \tag{3.88}$$

In particular for every $x' \in \mathcal{U}$ we have $\sigma_{x'}(\cdot) := \tilde{h}(\cdot,x') \in \mathcal{R}(x')$. Thus

$$\inf_{h_0\in\mathcal{P}(\mathcal{U})}\left\{\int\limits_S P_x\big(\tilde{h}(\cdot,x')\big)\,d\mathcal{H}^{N-1}(x)\right\} \geq \int\limits_S \left\{\inf_{\sigma\in\mathcal{R}(x')} P_x\big(\sigma(\cdot)\big)\right\} d\mathcal{H}^{N-1}(x)\,. \tag{3.89}$$

Therefore, we need only to prove the reverse inequality. Next observe that for every $x \in S$ we have

$$\inf_{\sigma\in\mathcal{R}(x')} P_x(\sigma) \leq P_x(0) := \int\limits_{\mathbb{R}}\int\limits_{I_1^{N-1}} \frac{1}{L}\times$$
$$F\bigg(p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{n}(x'),\Gamma(-s_1/L,x),\boldsymbol{B}\cdot v(x),\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)ds'ds_1$$
$$= \int\limits_{\mathbb{R}} F\bigg(p(t,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{n}(x'),\Gamma(t,x),\boldsymbol{B}\cdot v(x),\zeta\big(t,,f^-(x),f^+(x)\big)\bigg)\,dt \leq D\,, \tag{3.90}$$

where $D \in (0,+\infty)$ is a constant, not depending on x.

Next we prove that the function

$$\zeta(x) := \inf_{\sigma\in\mathcal{R}(x')} P_x(\sigma) \quad \forall x \in S \tag{3.91}$$

is Borel measurable. Indeed, consider $\mathcal{O}$, the subspace of the metric space $C^2_{loc}(\mathbb{R}^N,\mathbb{R}^d)$ (with the family of semi-norms $\{\|\cdot\|_{W^{2,\infty}(K)}\}_{K\subset\subset\mathbb{R}^N}$), containing all the functions $\bar{\sigma} \in C^\infty(\mathbb{R}^N,\mathbb{R}^d)$, that satisfy

$$\bar{\sigma}(y) = 0 \;\text{ if }\; |y_1| \geq 1/2, \;\text{ and }\; \bar{\sigma}(y+\boldsymbol{e}_j) = \bar{\sigma}(y) \;\; \forall j = 2,\ldots,N\,. \tag{3.92}$$

Then, since the metric space $C^2_{loc}(\mathbb{R}^N,\mathbb{R}^d)$ is separable, the subspace $\mathcal{O}$ is also separable and therefore there exists a countable subset $\mathcal{O}_r \subset \mathcal{O}$ which is dense in the topology of $C^2_{loc}(\mathbb{R}^N,\mathbb{R}^d)$. On the other hand for every $\bar{\sigma} \in \mathcal{O}$ and every $x' \in \mathcal{U}$ the function $\sigma_{x'} := \bar{\sigma}\big(y_1 - \nabla_{x'}g(x')\cdot y', y'\big)$ belongs to $\mathcal{R}(x')$ and moreover we have

$$\zeta(x) = \inf_{\sigma\in\mathcal{R}(x')} P_x(\sigma) = \inf_{\bar{\sigma}\in\mathcal{O}} P_x(\sigma_{x'}) = \inf_{\bar{\sigma}\in\mathcal{O}_r} P_x(\sigma_{x'})\,. \tag{3.93}$$

Thus since $\mathcal{O}_r$ is countable and since $P_x(\sigma_{x'})$ is Borel measurable on S, by (3.93) we deduce that $\zeta(x)$ is Borel measurable. Next fix any $\varepsilon > 0$. By Lusin's Theorem there exists a compact set $K \subset S$ such that $(\boldsymbol{A}\cdot\nabla v)^+(x)$, $(\boldsymbol{A}\cdot\nabla v)^-(x)$, $f^+(x)$, $f^-(x)$ and $\zeta(x)$ are continuous functions on K and

$$\mathcal{H}^{N-1}\big(S\setminus K\big) \leq \frac{\varepsilon}{2D}. \tag{3.94}$$

Here ζ is the function defined by (3.91). For any $x \in K$ there exists $\varphi_x \in \mathcal{R}(x')$ such that

$$P_x(\varphi_x) - \zeta(x) < \frac{\varepsilon}{4+4\mathcal{H}^{N-1}(S)}\,. \tag{3.95}$$

Then set

$$\gamma_{z,x}(y) := \varphi_x\bigg(y_1 + \big(\nabla_{z'}g(z') - \nabla_{x'}g(x')\big)\cdot y', y'\bigg). \tag{3.96}$$

Using the continuity by z of $\nabla_{z'}g(z')$, $\zeta(z)$ and $P_z(\gamma_{z,x})$ on K, we infer that for any $x \in K$ there exists $\delta_x > 0$ such that

$$P_z(\gamma_{z,x}) - \zeta(z) < \frac{\varepsilon}{2+2\mathcal{H}^{N-1}(S)}\,, \quad \forall z \in K\cap B_{\delta_x}(x)\,. \tag{3.97}$$

Since the set K is compact, there exists a finite number of points $x_1, x_2, \dots, x_l \in K$ such that $K \subset \bigcup_{j=1}^{l} B_{\delta_{x_j}}(x_j)$. Define the function $\bar{p}(y,x')$ on $\mathbb{R}^N \times \mathcal{U}$ by

$$\bar{p}(y,x') = \begin{cases} \gamma_{x,x_i}(y) & \forall x = (x_1, x') \in \big(K \bigcap B_{\delta_{x_i}}(x)\big) \setminus \bigcup_{1 \le j \le i-1} B_{\delta_{x_j}}(x_j) \qquad 1 \le i \le l, \\ 0 & \forall x \in S \setminus K\,. \end{cases} \tag{3.98}$$

Then from (3.94), (3.97) and (3.90) we get

$$\int_S P_x\big(\bar{p}(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x) - \int_S \zeta(x)\, d\mathcal{H}^{N-1}(x) < \varepsilon\,. \tag{3.99}$$

Moreover, $\bar{p}(y,x')$ satisfies

$$\bar{p}(y,x') = 0 \quad \text{if} \quad \big|\big(y \cdot \boldsymbol{q}_1(x')\big)\big| \ge \frac{1}{2}\boldsymbol{n}_1(x'), \quad \text{and} \quad \bar{p}\big(y + \boldsymbol{q}_j(x'), x'\big) = \bar{p}(y,x') \quad \forall j = 2,\dots,N\,, \tag{3.100}$$

and $\bar{p}(y,x') \in L^\infty\big(\mathcal{U}, C^k(K,\mathbb{R}^d)\big)$ for every $K \subset\subset \mathbb{R}^N$ and every natural k. Next define $p_0 : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^d$ by

$$p_0(y,x') := \bar{p}\Big(\big\{y_1 + \nabla_{x'} g(x') \cdot y', y'\big\}, x'\Big)\,, \tag{3.101}$$

Then

$$\bar{p}(y,x') := p_0\Big(\big\{y_1 - \nabla_{x'} g(x') \cdot y', y'\big\}, x'\Big)\,, \tag{3.102}$$

and

$$p_0(y,x') = 0 \quad \text{if} \quad |y_1| \ge 1/2, \quad \text{and} \quad p_0\big(y + \boldsymbol{e}_j, x'\big) = p_0(y,x') \quad \forall j = 2,\dots,N\,. \tag{3.103}$$

Moreover, $p_0(y,x') \in L^\infty\big(\mathcal{U}, C^k(K,\mathbb{R}^d)\big)$ for every $K \subset\subset \mathbb{R}^N$ and every natural k. Next let $\omega \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{R})$ be such that $\omega \ge 0$ and $\int_{\mathbb{R}^{N-1}} \omega(y')dy' = 1$. For any $0 < \rho < 1$ define

$$p_\rho(y,x') = \frac{1}{\rho^{N-1}} \int_{\mathbb{R}^{N-1}} \omega\Big(\frac{x'-z'}{\rho}\Big) p_0(y,z')\, dz' = \int_{\mathbb{R}^{N-1}} \omega(z') p_0(y, x' + \rho z')\, dz'\,. \tag{3.104}$$

Then $p_\rho \in C^\infty(\mathbb{R}^N \times \mathcal{U}, \mathbb{R}^d)$ and

$$p_\rho(y,x') = 0 \quad \text{if} \quad |y_1| \ge 1/2, \quad \text{and} \quad p_\rho\big(y + \boldsymbol{e}_j, x'\big) = p_\rho(y,x') \quad \forall j = 2,\dots,N\,. \tag{3.105}$$

Furthermore, there exists a constant $M > 0$, independent of ρ, y and x', such that for every $0 < \rho < 1$ we have

$$|\nabla_y^2 p_\rho(y,x')| + |\nabla_y p_\rho(y,x')| + |p_\rho(y,x')| \le M\,. \tag{3.106}$$

Moreover, for every y, for a.e. $x' \in \mathcal{U}$ we have

$$\nabla_y^2 p_\rho(y,x') \to \nabla_y^2 p_0(y,x')\,, \quad \nabla_y p_\rho(y,x') \to \nabla_y p_0(y,x')\,, \quad p_\rho(y,x') \to p_0(y,x') \quad \text{as} \quad \rho \to 0^+\,. \tag{3.107}$$

Therefore, if we define

$$\tilde{p}_\rho(y,x') := p_\rho\Big(\big\{y_1 - \nabla_{x'} g(x') \cdot y', y'\big\}, x'\Big)\,, \tag{3.108}$$

then, by (3.106) and (3.107)

$$\lim_{\rho \to 0^+} \int_S P_x\big(\tilde{p}_\rho(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x) = \int_S P_x\big(\bar{p}(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x)\,. \tag{3.109}$$

Thus, by (3.99) there exists $\rho_0 \in (0,1)$ such that

$$\int_S P_x\big(\tilde{p}_{\rho_0}(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x) - \int_S \Big\{\inf_{\sigma \in \mathcal{R}(x')} P_x\big(\sigma(\cdot)\big)\Big\}\, d\mathcal{H}^{N-1}(x) < 2\varepsilon\,. \tag{3.110}$$

Finally consider the sequence of compact subsets $K_n \subset\subset \mathcal{U}$, such that $K_n \subset K_{n+1}$ and $\bigcup_{n=1}^{+\infty} K_n = \mathcal{U}$. For every n consider $\xi_n(x') \in C_c^\infty(\mathcal{U},\mathbb{R})$, such that $\xi_n(x') = 1$ if $x' \in K_n$ and $0 \le \xi_n(x') \le 1$ for every $x' \in \mathcal{U}$. For every n define $h_n(y,x') := p_{\rho_0}(y,x')\xi_n(x')$. Then $h_n \in C^\infty(\mathbb{R}^N \times \mathcal{U},\mathbb{R}^d)$. Moreover, by (3.105) we have

$$h_n(y,x') = 0 \quad \text{if} \quad |y_1| \ge 1/2, \quad \text{and} \quad h_n\big(y + \boldsymbol{e}_j, x'\big) = h_n(y,x') \quad \forall j = 2,\dots,N\,, \tag{3.111}$$

and by the definition $\overline{\operatorname{supp} h_n} \subset \mathbb{R}^N \times \mathcal{U}$. Thus $h_n \in \mathcal{P}(\mathcal{U})$. Moreover, by (3.106), for every n we have

$$|\nabla_y^2 h_n(y,x')| + |\nabla_y h_n(y,x')| + |h_n(y,x')| \le M\,, \tag{3.112}$$

and by the definition for every y and x' we have

$$\nabla_y^2 h_n(y,x') \to \nabla_y^2 p_{\rho_0}(y,x')\,, \quad \nabla_y h_n(y,x') \to \nabla_y p_{\rho_0}(y,x')\,, \quad h_n(y,x') \to p_{\rho_0}(y,x') \quad \text{as} \quad n \to +\infty\,. \tag{3.113}$$

Thus if we set

$$\tilde{h}_n(y,x') := h_n\Big(\big\{y_1 - \nabla_{x'} g(x')\cdot y', y'\big\}, x'\Big)\,, \tag{3.114}$$

then, by (3.112) and (3.113)

$$\lim_{n\to+\infty} \int_S P_x\big(\tilde{h}_n(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x) = \int_S P_x\big(\tilde{p}_{\rho_0}(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x)\,. \tag{3.115}$$

Then, by (3.110) we obtain

$$\lim_{n\to+\infty} \int_S P_x\big(\tilde{h}_n(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x) - \int_S \Big\{\inf_{\sigma\in\mathcal{R}(x')} P_x\big(\sigma(\cdot)\big)\Big\}\, d\mathcal{H}^{N-1}(x) \le 2\varepsilon\,. \tag{3.116}$$

Therefore, since $\varepsilon > 0$ was chosen arbitrary and since $h_n \in \mathcal{P}(\mathcal{U})$ we deduce

$$\inf_{h_0\in\mathcal{P}(\mathcal{U})} \bigg\{\int_S P_x\big(\tilde{h}(\cdot,x')\big)\, d\mathcal{H}^{N-1}(x)\bigg\} \le \int_S \Big\{\inf_{\sigma\in\mathcal{R}(x')} P_x\big(\sigma(\cdot)\big)\Big\}\, d\mathcal{H}^{N-1}(x)\,, \tag{3.117}$$

which together with the reverse inequality (3.89) gives the desired result. $\square$

3.3 Construction of the approximating sequence in the general case

Lemma 3.3. *Let v, F, f, $\boldsymbol{A}$, $\boldsymbol{B}$, ψ_ε, $\boldsymbol{\nu}(x)$, p and Γ be the same as in Theorem 3.1. Then for every $\delta > 0$ there exist N Borel-measurable functions $\boldsymbol{p}_j(x) : J_{\boldsymbol{A}\cdot\nabla v} \to \mathbb{R}^N$ for $1 \le j \le N$, such that $\boldsymbol{p}_1(x) := \boldsymbol{\nu}(x)$, $\boldsymbol{p}_j(x)\cdot\boldsymbol{\nu}(x) = 0$ for $2 \le j \le N$ and $\{\boldsymbol{p}_j(x)\}_{1\le j\le N}$ is linearly independent system of vectors for every x, and there exists a sequence $\{v_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ such that $\lim_{\varepsilon\to0^+} \boldsymbol{A}\cdot\nabla v_\varepsilon = \boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+} \varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\} = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to0^+} \boldsymbol{B}\cdot v_\varepsilon = \boldsymbol{B}\cdot v$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+} (\boldsymbol{B}\cdot v_\varepsilon - \boldsymbol{B}\cdot v)/\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^k)$ for every $p \ge 1$ and*

$$0 \le \lim_{\varepsilon\to0} \int\limits_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\, \boldsymbol{A}\cdot\nabla v_\varepsilon,\, \boldsymbol{B}\cdot v_\varepsilon,\, f\Big)\, dx - \int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}} \bigg\{\inf_{\sigma\in\mathcal{Q}(x), L\in(0,1)} H_x\big(\sigma(\cdot), L\big)\bigg\}\, d\mathcal{H}^{N-1}(x) < \delta\,, \tag{3.118}$$

where

$$H_x(\sigma, L) := \int\limits_{I_N} \frac{1}{L} F\bigg(p(-s_1/L, x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\} \otimes \boldsymbol{\nu}(x) + L\nabla_y\{\boldsymbol{A}\cdot\nabla_y\sigma\}\big(T_x(s)\big),$$
$$\Gamma(-s_1/L, x) + \boldsymbol{A}\cdot\nabla_y\sigma\big(T_x(s)\big),\, \boldsymbol{B}\cdot v(x),\, \zeta\big(s_1, f^+(x), f^-(x)\big)\bigg)\, ds\,, \tag{3.119}$$

$$\mathcal{Q}(x) := \mathcal{D}_1\big(\boldsymbol{A}, \boldsymbol{\nu}(x), \boldsymbol{p}_2(x), \boldsymbol{p}_3(x), \dots, \boldsymbol{p}_N(x)\big) :=$$
$$\bigg\{\sigma \in C^\infty(\mathbb{R}^N,\mathbb{R}^d) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^d) \cap Lip(\mathbb{R}^N,\mathbb{R}^d) :\ \boldsymbol{A}\cdot\nabla\sigma(y) = 0 \ \ \textit{if} \ \ |y\cdot\boldsymbol{\nu}(x)| \ge 1/2$$
$$\textit{and} \ \ \sigma\big(y + \boldsymbol{p}_j(x)\big) = \sigma(y) \ \ \forall j = 2,3,\dots,N\bigg\}\,, \tag{3.120}$$

$$I_N := \Big\{ s \in \mathbb{R}^N : \ -1/2 < s_j < 1/2 \quad \forall j = 1, 2, \dots, N \Big\}, \tag{3.121}$$

and the linear transformation $T_x(s) = T_x(s_1, s_2, \dots, s_N) : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$T_x(s) := \sum_{j=1}^{N} s_j \, \boldsymbol{p}_j(x), \tag{3.122}$$

Moreover, $\boldsymbol{A} \cdot \nabla v_\varepsilon$, $\varepsilon \nabla \{\boldsymbol{A} \cdot \nabla v_\varepsilon\}$, $\boldsymbol{B} \cdot v_\varepsilon$ and $\nabla \{\boldsymbol{B} \cdot v_\varepsilon\}$ are bounded in L^∞ sequences, there exists a compact $K = K_\delta \subset\subset \Omega$ such that $v_\varepsilon(x) = \psi_\varepsilon(x)$ for every $0 < \varepsilon < 1$ and every $x \in \mathbb{R}^N \setminus K$, and

$$\limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \bigg(\int_\Omega \boldsymbol{A} \cdot \nabla v_\varepsilon(x)\, dx - \int_\Omega \{\boldsymbol{A} \cdot \nabla v\}(x)\, dx \bigg) \right| < +\infty, \tag{3.123}$$

Proof. First of all observe that

$$\int\limits_{\mathbb{R}} F\bigg(p(t,x)\Big\{ (\boldsymbol{A} \cdot \nabla v)^+(x) - (\boldsymbol{A} \cdot \nabla v)^-(x) \Big\} \otimes \boldsymbol{\nu}(x),\, \Gamma(t,x),\, \boldsymbol{B} \cdot v(x),\, \zeta\big(t, f^-(x), f^+(x)\big) \bigg)\, dt \le D_0, \tag{3.124}$$

where $D_0 \in (0, +\infty)$ is a constant, not depending on x.

Next since the set $\Omega \cap J_{\boldsymbol{A} \cdot \nabla v}$ is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set, oriented by $\boldsymbol{\nu}(x)$, $\Omega \cap J_{\boldsymbol{A} \cdot \nabla v}$ is σ-finite with respect to $\mathcal{H}^{N-1}$, there exist countably many C^1 hypersurfaces $\{S_k\}_{k=1}^\infty$ such that $\mathcal{H}^{N-1}\Big(\Omega \cap J_{\boldsymbol{A} \cdot \nabla v} \setminus \bigcup\limits_{k=1}^{\infty} S_k \Big) = 0$, and for $\mathcal{H}^{N-1}$-almost every $x \in \Omega \cap J_{\boldsymbol{A} \cdot \nabla v} \cap S_k$, $\boldsymbol{\nu}(x) = \boldsymbol{n}_k(x)$ where $\boldsymbol{n}_k(x)$ is a normal to S_k at the point x. We also may assume that for every $k \in \mathbb{N}$ we have $\mathcal{H}^{N-1}(S_k) < +\infty$, $\overline{S}_k \subset\subset \Omega$ and there exists a relabeling of the axes $\bar{x} := Z_k(x)$ such that for some function $g_k(x') \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ and a bounded open set $\mathcal{V}_k \subset \mathbb{R}^{N-1}$ we have

$$S_k = \{ x : \ Z_k(x) = \bar{x} = (\bar{x}_1, \bar{x}'),\ \bar{x}' \in \mathcal{V}_k,\ \bar{x}_1 = g_k(\bar{x}') \}. \tag{3.125}$$

Moreover $\boldsymbol{n}'_k(x) := Z_k\big(\boldsymbol{n}_k(x)\big) = (1, -\nabla_{\bar{x}'} g_k(\bar{x}'))/\sqrt{1 + |\nabla_{\bar{x}'} g_k(\bar{x}')|^2}$. Next clearly there exists $k_0 \in \mathbb{N}$ such that

$$\int_{(\Omega \cap J_{\boldsymbol{A} \cdot \nabla v} \setminus \cup_{k=1}^{k_0} S_k)} \Bigg\{ \\ \int\limits_{\mathbb{R}} F\bigg(p(t,x)\Big\{ (\boldsymbol{A} \cdot \nabla v)^+(x) - (\boldsymbol{A} \cdot \nabla v)^-(x) \Big\} \otimes \boldsymbol{\nu}(x), \Gamma(t,x), \boldsymbol{B} \cdot v(x), \zeta\big(t, f^-(x), f^+(x)\big) \bigg) dt \Bigg\}\, d\mathcal{H}^{N-1}(x) < \frac{\delta}{8}. \tag{3.126}$$

Next for every $k = 1, 2, \dots k_0$ there exists an open set $\mathcal{U}_k \subset \mathbb{R}^{N-1}$ such that if for every $k = 1, \dots k_0$ we set

$$S'_k := \{ x \in S_k : \ Z_k(x) = \bar{x} = (\bar{x}_1, \bar{x}'),\ \bar{x}' \in \mathcal{U}_k,\ \bar{x}_1 = g_k(\bar{x}') \}, \tag{3.127}$$

then $\overline{\mathcal{U}_k} \subset\subset \mathcal{V}_k$, $\overline{S'_k} \subset S_k \setminus (\cup_{j=1}^{k-1} \overline{S'_j})$ and

$$\mathcal{H}^{N-1}\bigg(\big\{ \cup_{j=1}^{k_0} S_j \big\} \setminus \big\{ \cup_{j=1}^{k_0} S'_j \big\} \bigg) < \frac{\delta}{8 D_0}. \tag{3.128}$$

In particular we have $\overline{S'_j} \cap \overline{S'_k} = \emptyset$ if $j \ne k$.

Next for every $j = 1, 2 \dots N$, for every $k = 1, 2, \dots k_0$ and every $x \in S'_k$ set

$$\boldsymbol{l}_j(x) := \begin{cases} \boldsymbol{n}'_k(x) = Z_k\big(\boldsymbol{n}_k(x)\big) & \text{if} \quad j = 1, \\ \big(\nabla_{\bar{x}} g_k(\bar{x}') \cdot \boldsymbol{e}_j\big)\, \boldsymbol{e}_1 + \boldsymbol{e}_j & \text{if} \quad 2 \le j \le N. \end{cases}, \tag{3.129}$$

where $\bar{x} := Z_k(x)$ (the corresponding to k relabeling of the axes) and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N\}$ is a standard orthonormal base in $\mathbb{R}^N$. Then there exist N Borel-measurable functions $\boldsymbol{p}_j(x) : J_{\boldsymbol{A} \cdot \nabla v} \to \mathbb{R}^N$ for $1 \le j \le N$, such that

$\boldsymbol{p}_1(x) := \boldsymbol{\nu}(x)$, $\boldsymbol{p}_j(x)\cdot\boldsymbol{\nu}(x) = 0$ for $2 \leq j \leq N$, $\{\boldsymbol{p}_j(x)\}_{1\leq j\leq N}$ is linearly independent system of vectors for every x and the following identity is satisfied

$$Z_k\big(\boldsymbol{p}_j(x)\big) = \begin{cases} \boldsymbol{l}_1(x) = Z_k\big(\boldsymbol{\nu}(x)\big) & \text{if } j = 1 \\ \sqrt{1 + \big|\nabla_{\bar{x}'} g_k\big(\bar{x}'\big)\big|^2} \cdot \boldsymbol{l}_j(x) & \text{if } j = 2, 3 \ldots N \end{cases} \qquad \forall x \in S'_k \quad \forall k = 1, 2, \ldots k_0\,, \tag{3.130}$$

where again $\bar{x} := Z_k(x)$. We let $\boldsymbol{p}_j(x)$ and $\boldsymbol{\nu}(x) := \boldsymbol{p}_1(x)$ be defined by (3.130) also for $x \in \cup_{k=1}^{k_0} S'_k \setminus J_{\boldsymbol{A}\cdot\nabla v}$

Next let

$$\mathcal{Q}_0(x) := \mathcal{D}_0\big(\boldsymbol{A}, \boldsymbol{\nu}(x), \boldsymbol{p}_2(x), \boldsymbol{p}_3(x), \ldots, \boldsymbol{p}_N(x)\big) := \Big\{\sigma \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) :\; \sigma(y) = 0 \;\text{ if }\; |y\cdot\boldsymbol{\nu}| \geq 1/2 \\ \text{and } \sigma\big(y + \boldsymbol{p}_j(x)\big) = \sigma(y) \;\; \forall j = 2, 3, \ldots, N\Big\}, \tag{3.131}$$

$\mathcal{Q}(x)$ be defined as in (3.120), $T_x(s)$ be as in (3.122) and $H_x(\sigma, L)$ be as in (3.119).

Observe that by the definition of $\sigma \in \mathcal{Q}(x)$ and by the properties of function p, for every $L \leq 1/2$, we have

$$H_x(\sigma, L) := \int\limits_{\mathbb{R}} \int\limits_{I_1^{N-1}} \frac{1}{L} F\bigg(p(-s_1/L, x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\} \otimes \boldsymbol{\nu}(x) + L\nabla_y\{\boldsymbol{A}\cdot\nabla_y\sigma\}\big(T_x(s)\big), \\ \Gamma(-s_1/L, x) + \boldsymbol{A}\cdot\nabla_y\sigma\big(T_x(s)\big),\, \boldsymbol{B}\cdot v(x),\, \zeta\big(s_1, f^+(x), f^-(x)\big)\bigg)\, ds' ds_1\,, \tag{3.132}$$

with $(s_1, s') := s \in \mathbb{R}\times\mathbb{R}^{N-1}$. On the other hand for every $\sigma \in \mathcal{Q}(x)$

$$\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma, L) \leq H_x(0, L) := \int\limits_{\mathbb{R}} \int\limits_{I_1^{N-1}} \frac{1}{L}\times \\ F\bigg(p(-s_1/L, x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\} \otimes \boldsymbol{\nu}(x),\, \Gamma(-s_1/L, x),\, \boldsymbol{B}\cdot v(x),\, \zeta\big(s_1, f^+(x), f^-(x)\big)\bigg)\, ds' ds_1 \\ = \int\limits_{\mathbb{R}} F\bigg(p(t, x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x) - (\boldsymbol{A}\cdot\nabla v)^-(x)\Big\} \otimes \boldsymbol{\nu}(x),\, \Gamma(t, x),\, \boldsymbol{B}\cdot v(x),\, \zeta\big(t, f^-(x), f^+(x)\big)\bigg)\, dt \leq D_0, \tag{3.133}$$

where $D_0 \in (0, +\infty)$ is a constant from (3.124), which doesn't depend on x and L. On the other hand by Lemma 2.2, for every $k = 1, 2, \ldots k_0$ and for every $x \in S'_k$ we deduce that

$$\inf_{L\in(0,1)} \left\{\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma, L)\right\} = \lim_{L\to 0^+} \left\{\inf_{\sigma\in\mathcal{Q}_0(x)} H_x\bigg(\sigma\,,\, L\sqrt{1 + |\nabla_{\bar{x}'} g_k(\bar{x}')|^2}\bigg)\right\}, \tag{3.134}$$

where $\bar{x} := Z_k(x)$. Therefore, by (3.134), (3.133) and the Dominated Convergence Theorem, we obtain

$$\int\limits_{\cup_{k=1}^{k_0} S'_k} \inf_{L\in(0,1)} \left\{\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma, L)\right\} d\mathcal{H}^{N-1}(x) = \\ \lim_{L\to 0^+} \int\limits_{\cup_{k=1}^{k_0} S'_k} \left\{\inf_{\sigma\in\mathcal{Q}_0(x)} H_x\bigg(\sigma\,,\, L\sqrt{1 + |\nabla_{\bar{x}'} g_k(\bar{x}')|^2}\bigg)\right\} d\mathcal{H}^{N-1}(x)\,, \tag{3.135}$$

and thus there exists $L_0 > 0$, such that $L_0\sqrt{1 + |\nabla_{\bar{x}'} g_k(\bar{x}')|^2} < 1$ and

$$\int\limits_{\cup_{k=1}^{k_0} S'_k} \left\{\inf_{\sigma\in\mathcal{Q}_0(x)} H_x\bigg(\sigma\,,\, L_0\sqrt{1 + |\nabla_{\bar{x}'} g_k(\bar{x}')|^2}\bigg)\right\} d\mathcal{H}^{N-1}(x) \\ - \int\limits_{\cup_{k=1}^{k_0} S'_k} \inf_{L\in(0,1)} \left\{\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma, L)\right\} d\mathcal{H}^{N-1}(x) < \frac{\delta}{8}\,, \tag{3.136}$$

On the other hand, changing the first variable of integration in (3.132), for every $x \in \cup_{k=1}^{k_0} S'_k$ we obtain

$$\inf_{\sigma\in\mathcal{Q}_0(x)} H_x\Big(\sigma\,,\,L\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}\Big) = \inf_{\sigma\in\mathcal{F}(x)} P_{x,k}(\sigma,L)\,, \tag{3.137}$$

where $P_{x,k}(\sigma,L)$ is defined by

$$P_{x,k}(\sigma,L) := \int_{\mathbb{R}}\int_{I_1^{N-1}} \frac{1}{L}F\bigg(p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{\nu}(x)+L\nabla_y\{\boldsymbol{A}\cdot\nabla_y\sigma\}\big(Q_{x,k}(s)\big),$$
$$\Gamma(-s_1/L,x)+\boldsymbol{A}\cdot\nabla_y\sigma\big(Q_{x,k}(s)\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)\,ds'ds_1\,, \tag{3.138}$$

with $(s_1,s') := s \in \mathbb{R}\times\mathbb{R}^{N-1}$ and the linear transformation $Q_{x,k}(s) = Q_{x,k}(s_1,s_2,\ldots,s_N) : \mathbb{R}^N \to \mathbb{R}^N$, defined by

$$Q_{x,k}(s) := s_1\boldsymbol{\nu}(x) + \sum_{j=2}^{N}\frac{s_j\boldsymbol{p}_j(x)}{\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}}\,, \tag{3.139}$$

and

$$\mathcal{F}(x) := \Bigg\{\sigma\in C^\infty(\mathbb{R}^N,\mathbb{R}^d):\ \sigma(y)=0 \ \text{ if } \ |y\cdot\boldsymbol{\nu}| \geq \frac{1}{2\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}}$$
$$\text{and }\ \sigma\bigg(y+\frac{\boldsymbol{p}_j(x)}{\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}}\bigg) = \sigma(y)\ \ \forall j=2,3,\ldots.N\Bigg\}. \tag{3.140}$$

Therefore, by (3.137) and by Lemma 3.2 we have

$$\sum_{k=1}^{k_0}\inf_{h_0\in\mathcal{P}(\mathcal{U}_k)}\bigg\{\int_{S'_k}P_{x,k}\big(\tilde{\lambda}_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x)\bigg\} =$$
$$\int_{\cup_{k=1}^{k_0}S'_k}\bigg\{\inf_{\sigma\in\mathcal{Q}_0(x)}H_x\Big(\sigma\,,\,L_0\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}\Big)\bigg\}\,d\mathcal{H}^{N-1}(x)\,, \tag{3.141}$$

where $\mathcal{P}(\mathcal{U})$ is a set of all functions $h_0(y,x') \in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$, satisfying

$$h_0(y,x') = 0 \ \text{ if } \ |y_1| \geq 1/2, \ \text{and} \ \ h_0(y+\boldsymbol{e}_j,x') = h_0(y,x') \ \ \forall j=2,\ldots,N\,, \tag{3.142}$$

and

$$\overline{\operatorname{supp} h_0(y,x')} \subset \mathbb{R}^N\times\mathcal{U}\,; \tag{3.143}$$

and $\tilde{\lambda}_k(y,x) \in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$ is given by

$$\tilde{\lambda}_k(y,x) := h_0\Big(\big\{\bar{y}_1-\nabla_{\bar{x}'}g_k(\bar{x}')\cdot\bar{y}',\bar{y}'\big\},\bar{x}'\Big)\,, \tag{3.144}$$

where again $\bar{x} := Z_k(x)$ and $\bar{y} = Z_k(y)$ (here Z_k is the appropriate relabeling). Thus for every $k = 1,2,\ldots k_0$ there exists $h_k \in \mathcal{P}(\mathcal{U}_k)$ such that

$$\int_{\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(\lambda_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x) \ - \int_{\cup_{k=1}^{k_0}S'_k}\bigg\{\inf_{\sigma\in\mathcal{Q}_0(x)}H_x\Big(\sigma\,,\,L_0\sqrt{1+|\nabla_{\bar{x}'}g_k(\bar{x}')|^2}\Big)\bigg\}\,d\mathcal{H}^{N-1}(x) \ < \ \frac{\delta}{8}\,, \tag{3.145}$$

where

$$\lambda_k(y,x) := h_k\Big(\big\{\bar{y}_1-\nabla_{\bar{x}'}g_k(\bar{x}')\cdot\bar{y}',\bar{y}'\big\},\bar{x}'\Big)\,. \tag{3.146}$$

Plugging (3.145) into (3.136) we deduce

$$\int\limits_{\cup_{k=1}^{k_0} S'_k} P_{x,k}\big(\lambda_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x) - \int\limits_{\cup_{k=1}^{k_0} S'_k} \inf_{L\in(0,1)}\left\{\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma,L)\right\}d\mathcal{H}^{N-1}(x) < \frac{\delta}{4}\,. \tag{3.147}$$

Next consider a radial function $\theta_0(z') = \kappa(|z'|) \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{R})$ such that $\operatorname{supp}\theta_0 \subset\subset B_1(0)$, $\theta_0 \geq 0$ and $\int_{\mathbb{R}^{N-1}} \theta_0(z')dz' = 1$. Then for any $\varepsilon > 0$ define the function $g_{k,\varepsilon}(x') : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$g_{k,\varepsilon}(x') := \frac{1}{\varepsilon^{N-1}}\int_{\mathbb{R}^{N-1}}\theta_0\Big(\frac{y'-x'}{\varepsilon}\Big)g_k(y')dy' = \int_{\mathbb{R}^{N-1}}\theta_0(z')g_k(x'+\varepsilon z')\,dz', \quad \forall x'\in\mathbb{R}^{N-1}\,. \tag{3.148}$$

Next set

$$h_k^{(L_0)}(y,x') := \frac{1}{L_0}h_k(L_0 y,x')\,, \tag{3.149}$$

then $h_k^{(L_0)} \in C^\infty(\mathbb{R}^N\times\mathbb{R}^{N-1},\mathbb{R}^d)$ satisfy

$$h_k^{(L_0)}(y,x') = 0 \ \text{ if } \ |y_1| \geq 1/(2L_0), \ \text{ and } \ h_k^{(L_0)}\big(y+(1/L_0)\boldsymbol{e}_j,x'\big) = h_k^{(L_0)}(y,x') \ \ \forall j=2,\ldots,N\,, \tag{3.150}$$

and

$$\overline{\operatorname{supp} h_k^{(L_0)}(y,x')} \subset \mathbb{R}^N\times\mathcal{U}_k\,. \tag{3.151}$$

For any $\varepsilon > 0$ define the function $\gamma_{k,\varepsilon}(x) \in C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ by

$$\gamma_{k,\varepsilon}(x) := \varepsilon h_k^{(L_0)}\left(\Big(\frac{\bar{x}_1 - g_{k,\varepsilon}(\bar{x}')}{\varepsilon},\frac{\bar{x}'}{\varepsilon}\Big),\bar{x}'\right), \tag{3.152}$$

where, as before, $\bar{x} := Z_k(x)$. Next clearly for every $k = 1,2\ldots,k_0$ there exists an open set $G_k \subset\subset \Omega$ such that for every k, $\overline{S'_k} \subset\subset G_k$ and $\overline{G_j}\cap\overline{G_k} = \emptyset$ if $j\neq k$. Thus there exists $\varepsilon_0\in(0,1)$ such that if $0<\varepsilon<\varepsilon_0$ then $\operatorname{supp} h_k^{(L_0)} \subset\subset G_k$. Therefore, for $0<\varepsilon<\varepsilon_0$ we can define $\gamma_\varepsilon(x)\in C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ by

$$\gamma_\varepsilon(x) := \begin{cases}\gamma_{k,\varepsilon}(x) & \text{if } \ x\in G_k \quad \forall k=1,2,\ldots k_0\,,\\ 0 & \text{otherwise.}\end{cases}\,, \tag{3.153}$$

Then we can set

$$v_\varepsilon(x) := \psi_\varepsilon(x) + \gamma_\varepsilon(x) \qquad \forall x\in\mathbb{R}^N\,. \tag{3.154}$$

Thus, as before in the conditions of Proposition 3.1, $\lim_{\varepsilon\to0^+}\boldsymbol{A}\cdot\nabla v_\varepsilon = \boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+}\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\} = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to0^+}\boldsymbol{B}\cdot v_\varepsilon = \boldsymbol{B}\cdot v$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+}(\boldsymbol{B}\cdot v_\varepsilon - \boldsymbol{B}\cdot v)/\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^k)$ for every $p\geq1$ and moreover, $\boldsymbol{A}\cdot\nabla v_\varepsilon$, $\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}$, $\boldsymbol{B}\cdot v_\varepsilon$ and $\nabla\{\boldsymbol{B}\cdot v_\varepsilon\}$ are bounded in L^∞ sequences, there exists a compact $K\subset\subset\Omega$ such that $v_\varepsilon(x) = \psi_\varepsilon(x)$ for every $0<\varepsilon<1$ and every $x\in\mathbb{R}^N\setminus K$ and we have (3.123).

Next clearly, for $0<\varepsilon<\varepsilon_0$ we have

$$\int\limits_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx - \int\limits_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)\,dx$$
$$= \sum_{k=1}^{k_0}\left\{\int\limits_{G_k}\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx - \int\limits_{G_k}\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)\,dx\right\}$$
$$= \sum_{k=1}^{k_0}\left\{\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\big\{\boldsymbol{A}\cdot\nabla(\psi_\varepsilon+\gamma_{k,\varepsilon})\big\},\,\boldsymbol{A}\cdot\nabla(\psi_\varepsilon+\gamma_{k,\varepsilon}),\,\boldsymbol{B}\cdot(\psi_\varepsilon+\gamma_{k,\varepsilon}),\,f\Big)\,dx\right.$$
$$\left. - \int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot\psi_\varepsilon,\,f\Big)\,dx\right\}. \tag{3.155}$$

On the other hand, by Proposition 3.1, for every $k = 1, 2 \dots, k_0$

$$\lim_{\varepsilon\to 0^+}\left\{\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\big\{\boldsymbol{A}\cdot\nabla(\psi_\varepsilon+\gamma_{k,\varepsilon})\big\},\,\boldsymbol{A}\cdot\nabla(\psi_\varepsilon+\gamma_{k,\varepsilon}),\,\boldsymbol{B}\cdot(\psi_\varepsilon+\gamma_{k,\varepsilon}),\,f\Big)\,dx \right.$$
$$\left. -\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx\right\} = \int_{S'_k} P_{x,k}\big(\lambda_k(\cdot,x)\big)\,d\mathcal{H}^{N-1}(x) - \int_{S'_k} P_{x,k}\big(0\big)\,d\mathcal{H}^{N-1}(x)\,. \tag{3.156}$$

Plugging (3.156) into (3.155) we obtain

$$\lim_{\varepsilon\to 0^+}\left\{\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx - \int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx\right\} =$$
$$\int_{\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(\lambda_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x) - \int_{\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(0,L_0\big)\,d\mathcal{H}^{N-1}(x)\,. \tag{3.157}$$

On the other hand by Theorem 3.1, (3.133) and (3.138) we have

$$\lim_{\varepsilon\to 0^+}\left\{\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla\psi_\varepsilon\},\,\boldsymbol{A}\cdot\nabla\psi_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx\right\} = \int_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}} P_{x,k}\big(0,L_0\big)\,d\mathcal{H}^{N-1}(x)\,. \tag{3.158}$$

Thus combining (3.158) and (3.157) together with (3.133), (3.138) and the fact that $P_{x,k}(0,L) = 0$ if $x \notin \Omega\cap J_{\boldsymbol{A}\cdot\nabla v}$, we obtain

$$\lim_{\varepsilon\to 0^+}\left\{\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx\right\} =$$
$$\int_{\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(\lambda_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x) + \int_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}\setminus\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(0,L_0\big)\,d\mathcal{H}^{N-1}(x) =$$
$$\int_{\cup_{k=1}^{k_0}S'_k} P_{x,k}\big(\lambda_k(\cdot,x),L_0\big)\,d\mathcal{H}^{N-1}(x) + \int_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}\setminus\cup_{k=1}^{k_0}S'_k}\Bigg\{$$
$$\int_{\mathbb{R}} F\bigg(p(t,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{\nu}(x),\,\Gamma(t,x),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(t,f^-(x),f^+(x)\big)\bigg)\,dt\Bigg\}\,d\mathcal{H}^{N-1}(x)\,. \tag{3.159}$$

Therefore, by (3.126), (3.147) and (3.159) we obtain

$$\lim_{\varepsilon\to 0^+}\left\{\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\boldsymbol{A}\cdot\nabla v_\varepsilon,\boldsymbol{B}\cdot v_\varepsilon,f\Big)dx\right\} < \frac{3\delta}{8} + \int_{\cup_{k=1}^{k_0}S'_k}\inf_{L\in(0,1)}\left\{\inf_{\sigma\in\mathcal{Q}(x)} H_x(\sigma,L)\right\}d\mathcal{H}^{N-1}(x) +$$
$$\int_{\cup_{k=1}^{k_0}S_k\setminus\cup_{k=1}^{k_0}S'_k}\left\{\int_{\mathbb{R}} F\bigg(p(t,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{\nu}(x),\Gamma(t,x),\boldsymbol{B}\cdot v(x),\zeta\big(t,f^-(x),f^+(x)\big)\bigg)dt\right.$$
$$\left.\vphantom{\int}\right\}\,d\mathcal{H}^{N-1}(x)\,. \tag{3.160}$$

Finally plugging (3.128) and (3.133) into (3.160) we infer

$$\lim_{\varepsilon\to 0^+}\left\{\int\limits_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx\right\}<$$

$$\int\limits_{\cup_{k=1}^{k_0}S'_k}\inf_{L\in(0,1)}\left\{\inf_{\sigma\in\mathcal{Q}(x)}H_x(\sigma,L)\right\}d\mathcal{H}^{N-1}(x)+\frac{\delta}{2}\leq\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{\sigma\in\mathcal{Q}(x),L\in(0,1)}H_x(\sigma,L)\right\}d\mathcal{H}^{N-1}(x)+\frac{\delta}{2}\,, \tag{3.161}$$

where the last inequality we infer since if $x\notin\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}$ then again by (3.133) we have

$$0\leq\inf_{\sigma\in\mathcal{Q}(x)}H_x(\sigma,L)\leq H_x(0,L)=0\,.$$

This completes the proof. $\square$

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^N$ be an open set. Furthermore, let $\boldsymbol{A}\in\mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$, $\boldsymbol{B}\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^k)$ and let $F\in C^1(\mathbb{R}^{m\times N}\times\mathbb{R}^m\times\mathbb{R}^k\times\mathbb{R}^q,\mathbb{R})$, satisfying $F\geq 0$. Let $f\in BV_{loc}(\mathbb{R}^N,\mathbb{R}^q)\cap L^\infty$ and $v\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d)$ be such that $\boldsymbol{A}\cdot\nabla v\in BV(\mathbb{R}^N,\mathbb{R}^m)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^m)$ and $\boldsymbol{B}\cdot v\in Lip(\mathbb{R}^N,\mathbb{R}^k)\cap W^{1,1}(\mathbb{R}^N,\mathbb{R}^k)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^k)$, $\|D(\boldsymbol{A}\cdot\nabla v)\|(\partial\Omega)=0$ and $F\big(0,\{\boldsymbol{A}\cdot\nabla v\}(x),\{\boldsymbol{B}\cdot v\}(x),f(x)\big)=0$ a.e. in Ω. Moreover, assume that for $\mathcal{H}^{N-1}$-a.e. $x\in J_{\boldsymbol{A}\cdot\nabla v}\cap\Omega$ there exists a distribution $\gamma_x(\cdot)\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d)$ such that*

$$\{\boldsymbol{A}\cdot\nabla\gamma_x\}(z)=\begin{cases}(\boldsymbol{A}\cdot\nabla v)^+(x) & \text{if}\quad z\cdot\boldsymbol{\nu}(x)\geq 0\,,\\ (\boldsymbol{A}\cdot\nabla v)^-(x) & \text{if}\quad z\cdot\boldsymbol{\nu}(x)<0\end{cases} \tag{3.162}$$

(with $\boldsymbol{\nu}(x)$ denoting the orientation vector of $J_{\boldsymbol{A}\cdot\nabla v}$). Finally we assume that for $\mathcal{H}^{N-1}$-a.e. $x\in J_{\boldsymbol{A}\cdot\nabla v}\cap\Omega$ for every system $\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\}$ of linearly independent vectors in $\mathbb{R}^N$ satisfying $\boldsymbol{k}_1(x)=\boldsymbol{\nu}(x)$ and $\boldsymbol{k}_j(x)\cdot\boldsymbol{\nu}(x)=0$ for $j\geq 2$, and for every $\xi(z)\in\mathcal{W}_x(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_N)$ there exists $\zeta(z)\in\mathcal{W}'_x(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_N)$ such that $\boldsymbol{A}\cdot\nabla\xi(z)\equiv\boldsymbol{A}\cdot\nabla\zeta(z)$, where

$$\mathcal{W}_x\big(\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\big):=\Big\{u\in C^\infty(\mathbb{R}^N,\mathbb{R}^d):\ \boldsymbol{A}\cdot\nabla u(y)=0\ \text{ if }\ |y\cdot\boldsymbol{\nu}(x)|\geq 1/2,$$
$$\text{and}\ \ \boldsymbol{A}\cdot\nabla u\big(y+\boldsymbol{k}_j(x)\big)=\boldsymbol{A}\cdot\nabla u(y)\ \ \forall j=2,3,\dots,N\Big\}, \tag{3.163}$$

and

$$\mathcal{W}'_x\big(\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\big):=$$
$$\Big\{u\in C^\infty(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)\cap Lip(\mathbb{R}^N,\mathbb{R}^d):\ \boldsymbol{A}\cdot\nabla u(y)=0\ \text{ if }\ |y\cdot\boldsymbol{\nu}(x)|\geq 1/2$$
$$\text{and}\ \ u\big(y+\boldsymbol{k}_j(x)\big)=u(y)\ \ \forall j=2,3,\dots,N\Big\}. \tag{3.164}$$

Then, for $\eta\in\mathcal{V}_0$, for every $\delta>0$ there exist a sequence $\{v_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ such that $\lim_{\varepsilon\to 0^+}\boldsymbol{A}\cdot\nabla v_\varepsilon=\boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to 0^+}\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}=0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to 0^+}\boldsymbol{B}\cdot v_\varepsilon=\boldsymbol{B}\cdot v$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to 0^+}(\boldsymbol{B}\cdot v_\varepsilon-\boldsymbol{B}\cdot v)/\varepsilon=0$ in $L^p(\mathbb{R}^N,\mathbb{R}^k)$ for every $p\geq 1$ and

$$0\leq\lim_{\varepsilon\to 0}\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx$$
$$-\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{L>0}\left(\inf_{\sigma\in\mathcal{W}_0(x,\boldsymbol{k}_1,\dots,\boldsymbol{k}_N)}E_x\big(\sigma(\cdot),L\big)\right)\right\}d\mathcal{H}^{N-1}(x)<\delta\,, \tag{3.165}$$

where

$$E_x\big(\sigma(\cdot),L\big)\ :=\int\limits_{I_N}\frac{1}{L}F\Big(L\nabla\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\Big)\,ds\,, \tag{3.166}$$

$$\zeta(s,a,b):=\begin{cases} a & \textit{if } s>0\,,\\ b & \textit{if } s<0\,,\end{cases} \tag{3.167}$$

$$\mathcal{W}_0\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\big):= \Big\{u\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d):\boldsymbol{A}\cdot\nabla u\in C^1(\mathbb{R}^N,\mathbb{R}^m),\;\; \boldsymbol{A}\cdot\nabla u(y)=(\boldsymbol{A}\cdot\nabla v)^-(x)\;\;\textit{if}\;\; y\cdot\boldsymbol{\nu}(x)\le -1/2,$$
$$\boldsymbol{A}\cdot\nabla u(y)=(\boldsymbol{A}\cdot\nabla v)^+(x)\;\;\textit{if}\;\; y\cdot\boldsymbol{\nu}(x)\ge 1/2\;\;\textit{and}\;\;\boldsymbol{A}\cdot\nabla u\big(y+\boldsymbol{k}_j(x)\big)=\boldsymbol{A}\cdot\nabla u(y)\;\;\forall j=2,3,\dots,N\Big\}, \tag{3.168}$$

$$I_N:=\Big\{s\in\mathbb{R}^N:\;-1/2<s_j<1/2\quad\forall j=1,2,\dots,N\Big\}, \tag{3.169}$$

$\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\}$ is an arbitrary system of linearly independent vectors in $\mathbb{R}^N$ satisfying $\boldsymbol{k}_1(x)=\boldsymbol{\nu}(x)$ and $\boldsymbol{k}_j(x)\cdot\boldsymbol{\nu}(x)=0$ for $j\ge 2$, the linear transformation $S_x(s)=S_x(s_1,s_2,\dots,s_N):\mathbb{R}^N\to\mathbb{R}^N$ is defined by

$$S_x(s):=\sum_{j=1}^N s_j\,\boldsymbol{k}_j(x)\,, \tag{3.170}$$

and we assume that the orientation of J_f coincides with the orientation of $J_{\boldsymbol{A}\cdot\nabla v}$ $\mathcal{H}^{N-1}$ a.e. on $J_f\cap J_{\boldsymbol{A}\cdot\nabla v}$. Moreover, $\boldsymbol{A}\cdot\nabla v_\varepsilon$, $\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}$, $\boldsymbol{B}\cdot v_\varepsilon$ and $\nabla\{\boldsymbol{B}\cdot v_\varepsilon\}$ are bounded in L^∞ sequences, there exists a compact $K=K_\delta\subset\subset\Omega$ such that $v_\varepsilon(x)=\psi_\varepsilon(x)$ for every $0<\varepsilon<1$ and every $x\in\mathbb{R}^N\setminus K$, where

$$\psi_\varepsilon(x):=\frac{1}{\varepsilon^N}\Big\langle\eta\Big(\frac{y-x}{\varepsilon}\Big),v(y)\Big\rangle, \tag{3.171}$$

and

$$\limsup_{\varepsilon\to0^+}\left|\frac{1}{\varepsilon}\bigg(\int_\Omega\boldsymbol{A}\cdot\nabla v_\varepsilon(x)\,dx-\int_\Omega\{\boldsymbol{A}\cdot\nabla v\}(x)\,dx\bigg)\right|<+\infty\,, \tag{3.172}$$

Proof. Let η and ψ_ε be the same as in Theorem 3.1 and

$$\Gamma(t,x):=\bigg(\int_{-\infty}^t p(s,x)ds\bigg)\{\boldsymbol{A}\cdot\nabla v\}^-(x)+\bigg(\int_t^{+\infty}p(s,x)ds\bigg)\{\boldsymbol{A}\cdot\nabla v\}^+(x)\,, \tag{3.173}$$

where $p(t,x)$ is defined by

$$p(t,x):=\int_{H^0_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+y)\,d\mathcal{H}^{N-1}(y)\,. \tag{3.174}$$

Then, by Lemma 3.3 and Lemma 2.1, for every $\delta>0$ there exist N Borel-measurable functions $\boldsymbol{p}_j(x):J_{\boldsymbol{A}\cdot\nabla v}\to\mathbb{R}^N$ for $1\le j\le N$, such that $\boldsymbol{p}_1(x):=\boldsymbol{\nu}(x)$, $\boldsymbol{p}_j(x)\cdot\boldsymbol{\nu}(x)=0$ for $2\le j\le N$ and $\{\boldsymbol{p}_j(x)\}_{1\le j\le N}$ is linearly independent system of vectors for every x, and there exists a sequence $\{v_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ such that $\lim_{\varepsilon\to0^+}\boldsymbol{A}\cdot\nabla v_\varepsilon=\boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+}\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}=0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$, $\lim_{\varepsilon\to0^+}\boldsymbol{B}\cdot v_\varepsilon=\boldsymbol{B}\cdot v$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+}(\boldsymbol{B}\cdot v_\varepsilon-\boldsymbol{B}\cdot v)/\varepsilon=0$ in $L^p(\mathbb{R}^N,\mathbb{R}^k)$ for every $p\ge1$ and

$$0\le\lim_{\varepsilon\to0}\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\Big)\,dx-\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\bigg\{\inf_{\sigma\in\mathcal{Q}(x),L\in(0,1)}H_x\big(\sigma(\cdot),L\big)\bigg\}\,d\mathcal{H}^{N-1}(x)<\delta\,, \tag{3.175}$$

where

$$H_x(\sigma,L):=\int\limits_{I_N}\frac{1}{L}F\bigg(p(-s_1/L,x)\Big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\Big\}\otimes\boldsymbol{\nu}(x)+L\nabla\{\boldsymbol{A}\cdot\nabla\sigma\}\big(T_x(s)\big),$$
$$\Gamma(-s_1/L,x)+\boldsymbol{A}\cdot\nabla\sigma\big(T_x(s)\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)\,ds\,, \tag{3.176}$$

$$\mathcal{Q}(x) := \mathcal{D}_1\big(\boldsymbol{A}, \boldsymbol{\nu}(x), \boldsymbol{p}_2(x), \boldsymbol{p}_3(x), \ldots, \boldsymbol{p}_N(x)\big) := \\ \bigg\{\sigma \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap Lip(\mathbb{R}^N, \mathbb{R}^d) : \ \boldsymbol{A} \cdot \nabla \sigma(y) = 0 \ \text{ if } \ |y \cdot \boldsymbol{\nu}(x)| \geq 1/2 \\ \text{and } \ \sigma\big(y + \boldsymbol{p}_j(x)\big) = \sigma(y) \ \ \forall j = 2, 3, \ldots, N\bigg\}, \tag{3.177}$$

and the linear transformation $T_x(s) = T_x(s_1, s_2, \ldots, s_N) : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$T_x(s) := \sum_{j=1}^N s_j \, \boldsymbol{p}_j(x)\,, \tag{3.178}$$

Moreover, $\boldsymbol{A} \cdot \nabla v_\varepsilon$, $\varepsilon \nabla \{\boldsymbol{A} \cdot \nabla v_\varepsilon\}$, $\boldsymbol{B} \cdot v_\varepsilon$ and $\nabla\{\boldsymbol{B} \cdot v_\varepsilon\}$ are bounded in L^∞ sequences, there exists a compact $K \subset\subset \Omega$ such that $v_\varepsilon(x) = \psi_\varepsilon(x)$ for every $0 < \varepsilon < 1$ and every $x \in \mathbb{R}^N \setminus K$ and we have (3.172).

So we only need to prove that

$$\int\limits_{\Omega \cap J_{\boldsymbol{A}\cdot\nabla v}} \inf_{L>0} \left(\inf_{\sigma \in \mathcal{W}_0(x, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)} E_x\big(\sigma(\cdot), L\big)\right) d\mathcal{H}^{N-1}(x) \ = \int\limits_{\Omega \cap J_{\boldsymbol{A}\cdot\nabla v}} \left\{ \inf_{\sigma \in \mathcal{Q}(x), L \in (0,1)} H_x\big(\sigma(\cdot), L\big)\right\} d\mathcal{H}^{N-1}(x), \tag{3.179}$$

for any choice of the system $\{\boldsymbol{k}_1(x), \boldsymbol{k}_2(x), \ldots \boldsymbol{k}_N(x)\}$ of linearly independent vectors in $\mathbb{R}^N$ satisfying $\boldsymbol{k}_1(x) = \boldsymbol{\nu}(x)$ and $\boldsymbol{k}_j(x) \cdot \boldsymbol{\nu}(x) = 0$ for $j \geq 2$. By Proposition 2.1 we have that the left hand side of (3.179) is independent on the choice of the system $\{\boldsymbol{k}_1(x), \boldsymbol{k}_2(x), \ldots \boldsymbol{k}_N(x)\}$. Therefore, from now we may assume that $\boldsymbol{k}_j(x) = \boldsymbol{p}_j(x)$ for every x and j. Thus in particular $S_x = T_x$ and $\mathcal{Q}(x) = \mathcal{W}'_x(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \boldsymbol{k}_N)$. On the other hand, for $\mathcal{H}^{N-1}$-a.e. $x \in J_{\boldsymbol{A}\cdot\nabla v} \cap \Omega$ we have $\xi(z) \in \mathcal{W}_x(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \boldsymbol{k}_N)$ there exists $\zeta(z) \in \mathcal{W}'_x(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \boldsymbol{k}_N)$ such that $\boldsymbol{A} \cdot \nabla \xi(z) \equiv \boldsymbol{A} \cdot \nabla \zeta(z)$, and therefore we have

$$\int\limits_{\Omega \cap J_{\boldsymbol{A}\cdot\nabla v}} \inf_{\sigma \in \mathcal{Q}(x), L \in (0,1)} H_x\big(\sigma(\cdot), L\big)\, d\mathcal{H}^{N-1}(x) \ = \int\limits_{\Omega \cap J_{\boldsymbol{A}\cdot\nabla v}} \inf_{L \in (0,1)} \left(\inf_{\sigma \in \mathcal{W}_x(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \boldsymbol{k}_N)} H_x\big(\sigma(\cdot), L\big)\right) d\mathcal{H}^{N-1}(x). \tag{3.180}$$

Next for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap J_{\boldsymbol{A}\cdot\nabla v}$ there exists a distribution $\gamma_x(\cdot) \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^d)$ such that we have (3.162). For any $\varepsilon > 0$ and any fixed $z \in \mathbb{R}^N$ set

$$\bar{\gamma}_x(z) := \Big\langle \eta(y - z), \gamma_x(y) \Big\rangle \tag{3.181}$$

(see notations and definitions in the beginning of the paper). Then $\bar{\gamma}_x(z) \in C^\infty(\mathbb{R}^N, \mathbb{R}^d)$. Moreover clearly

$$\boldsymbol{A} \cdot \{\nabla \bar{\gamma}_x(z)\} = \int_{\mathbb{R}^N} \eta(y - z) \cdot \{\boldsymbol{A} \cdot \nabla \gamma_x\}(y)\, dy = \int_{\mathbb{R}^N} \eta(y) \cdot \big\{\boldsymbol{A} \cdot \nabla \gamma_x\big\}(z + y)\, dy. \tag{3.182}$$

Plugging it into (3.162) we deduce

$$\boldsymbol{A} \cdot \{\nabla \bar{\gamma}_x(z)\} = \Gamma\big(- z \cdot \boldsymbol{\nu}(x), x\big)\,. \tag{3.183}$$

Thus for every $L \in (0, 1)$, the function $\bar{\gamma}_{x,L}(z) := L\bar{\gamma}_x(z/L)$ belongs to $\mathcal{W}'_0\big(x, \boldsymbol{k}_1(x), \boldsymbol{k}_2(x), \ldots \boldsymbol{k}_N(x)\big)$ where

$$\mathcal{W}'_0\big(x, \boldsymbol{k}_1(x), \boldsymbol{k}_2(x), \ldots \boldsymbol{k}_N(x)\big) := \\ \bigg\{u \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) : \ \ \boldsymbol{A} \cdot \nabla u(y) = (\boldsymbol{A} \cdot \nabla v)^-(x) \ \text{ if } \ y \cdot \boldsymbol{\nu}(x) \leq -1/2, \\ \boldsymbol{A} \cdot \nabla u(y) = (\boldsymbol{A} \cdot \nabla v)^-(x) \ \text{ if } \ y \cdot \boldsymbol{\nu}(x) \geq 1/2 \ \text{ and } \ \boldsymbol{A} \cdot \nabla u\big(y + \boldsymbol{k}_j(x)\big) = \boldsymbol{A} \cdot \nabla u(y) \ \ \forall j \geq 2\bigg\}. \tag{3.184}$$

Moreover, $\boldsymbol{A}\cdot\{\nabla\bar{\gamma}_{x,L}\}\big(T_x(s)\big)=\Gamma\big(-s_1/L,x\big)$ and $\nabla\big(\boldsymbol{A}\cdot\{\nabla\bar{\gamma}_{x,L}\}\big)\big(T_x(s)\big)=p(-s_1/L,x)\big\{(\boldsymbol{A}\cdot\nabla v)^+(x)-(\boldsymbol{A}\cdot\nabla v)^-(x)\big\}\otimes\boldsymbol{\nu}(x)$. Thus

$$\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{L\in(0,1)}\left(\inf_{\sigma\in\mathcal{W}_x(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_N)}H_x\big(\sigma(\cdot),L\big)\right)\right\}d\mathcal{H}^{N-1}(x)$$
$$=\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{L\in(0,1)}\left(\inf_{\sigma\in\mathcal{W}'_0(x,\boldsymbol{k}_1,\dots,\boldsymbol{k}_N)}E_x\big(\sigma(\cdot),L\big)\right)\right\}d\mathcal{H}^{N-1}(x)\,,\quad(3.185)$$

and plugging it into (3.180) we obtain

$$\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{\sigma\in\mathcal{Q}(x),L\in(0,1)}H_x\big(\sigma(\cdot),L\big)\right\}d\mathcal{H}^{N-1}(x)$$
$$=\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{L\in(0,1)}\left(\inf_{\sigma\in\mathcal{W}'_0(x,\boldsymbol{k}_1,\dots,\boldsymbol{k}_N)}E_x\big(\sigma(\cdot),L\big)\right)\right\}d\mathcal{H}^{N-1}(x)\,,\quad(3.186)$$

Finally, using (3.186) and applying Lemma 2.1 we obtain (3.179). □

By the same method we can prove the following more general result.

Theorem 3.3. *Let $\Omega\subset\mathbb{R}^N$ be an open set. Furthermore, let $\boldsymbol{A}\in\mathcal{L}(\mathbb{R}^{d\times N};\mathbb{R}^m)$, $\boldsymbol{B}\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^k)$ and let $F\in C^1\big(\mathbb{R}^{m\times N^n}\times\mathbb{R}^{m\times N^{(n-1)}}\times\ldots\times\mathbb{R}^{m\times N}\times\mathbb{R}^m\times\mathbb{R}^k\times\mathbb{R}^q,\mathbb{R}\big)$, satisfying $F\geq0$. Let $f\in BV_{loc}(\mathbb{R}^N,\mathbb{R}^q)\cap L^\infty$ and $v\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d)$ be such that $\boldsymbol{A}\cdot\nabla v\in BV(\mathbb{R}^N,\mathbb{R}^m)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^m)$ and $\boldsymbol{B}\cdot v\in Lip(\mathbb{R}^N,\mathbb{R}^k)\cap W^{1,1}(\mathbb{R}^N,\mathbb{R}^k)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^k)$, $\|D(\boldsymbol{A}\cdot\nabla v)\|(\partial\Omega)=0$ and*

$$F\Big(0,0,\dots,0,\{\boldsymbol{A}\cdot\nabla v\}(x),\{\boldsymbol{B}\cdot v\}(x),f(x)\Big)=0\quad a.e.\ in\ \Omega\,.$$

Moreover, assume that for $\mathcal{H}^{N-1}$-a.e. $x\in\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}$ there exists a distribution $\gamma_x(\cdot)\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d)$ such that

$$\{\boldsymbol{A}\cdot\nabla\gamma_x\}(z)=\begin{cases}(\boldsymbol{A}\cdot\nabla v)^+(x) & \text{if}\quad z\cdot\boldsymbol{\nu}(x)\geq0\,,\\(\boldsymbol{A}\cdot\nabla v)^-(x) & \text{if}\quad z\cdot\boldsymbol{\nu}(x)<0\end{cases}\quad(3.187)$$

(with $\boldsymbol{\nu}(x)$ denoting the orientation vector of $J_{\boldsymbol{A}\cdot\nabla v}$). Finally we assume that for $\mathcal{H}^{N-1}$-a.e. $x\in\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}$ for every system $\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\}$ of linearly independent vectors in $\mathbb{R}^N$ satisfying $\boldsymbol{k}_1(x)=\boldsymbol{\nu}(x)$ and $\boldsymbol{k}_j(x)\cdot\boldsymbol{\nu}(x)=0$ for $j\geq2$, and for every $\xi(z)\in\mathcal{W}_{(x,n)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_N)$ there exists $\zeta(z)\in\mathcal{W}'_{(x,n)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_N)$ such that $\boldsymbol{A}\cdot\nabla\xi(z)\equiv\boldsymbol{A}\cdot\nabla\zeta(z)$, where

$$\mathcal{W}_{(x,n)}\big(\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\big):=\Big\{u\in C^\infty(\mathbb{R}^N,\mathbb{R}^d):\ \boldsymbol{A}\cdot\nabla u(y)=0\ \text{ if }\ |y\cdot\boldsymbol{\nu}(x)|\geq1/2,$$
$$\text{and}\ \ \boldsymbol{A}\cdot\nabla u\big(y+\boldsymbol{k}_j(x)\big)=\boldsymbol{A}\cdot\nabla u(y)\ \ \forall j=2,3,\dots,N\Big\},\quad(3.188)$$

and

$$\mathcal{W}'_{(x,n)}\big(\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\dots\boldsymbol{k}_N(x)\big):=$$
$$\Big\{u\in C^\infty(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d):\ \nabla^j u\in L^\infty(\mathbb{R}^N,\mathbb{R}^{d\times N^j})\ \text{for}\ j\leq n\,,$$
$$\boldsymbol{A}\cdot\nabla u(y)=0\ \text{ if }\ |y\cdot\boldsymbol{\nu}(x)|\geq1/2\ \text{ and }\ u\big(y+\boldsymbol{k}_j(x)\big)=u(y)\ \ \forall j=2,3,\dots,N\Big\}.\quad(3.189)$$

Then, for $\eta\in\mathcal{V}_0$, for every $\delta>0$ there exist a sequence $\{v_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ such that $\lim_{\varepsilon\to0^+}\boldsymbol{A}\cdot\nabla v_\varepsilon=\boldsymbol{A}\cdot\nabla v$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+}\varepsilon^j\nabla^j\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}=0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N^j})$ for every $j=1,2,\dots n$, $\lim_{\varepsilon\to0^+}\boldsymbol{B}\cdot v_\varepsilon=$

$\boldsymbol{B}\cdot v$ *in* $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+}(\boldsymbol{B}\cdot v_\varepsilon-\boldsymbol{B}\cdot v)/\varepsilon=0$ *in* $L^p(\mathbb{R}^N,\mathbb{R}^k)$ *for every* $p\geq1$ *and*

$$0\leq\lim_{\varepsilon\to0}\int\limits_{\Omega}\frac{1}{\varepsilon}F\bigg(\varepsilon^n\nabla^n\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\varepsilon^{n-1}\nabla^{n-1}\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\ldots,\,\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\},\,\boldsymbol{A}\cdot\nabla v_\varepsilon,\,\boldsymbol{B}\cdot v_\varepsilon,\,f\bigg)\,dx\\-\int\limits_{\Omega\cap J_{\boldsymbol{A}\cdot\nabla v}}\left\{\inf_{L>0}\left(\inf_{\sigma\in\mathcal{W}_0^{(n)}(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)}E_{x,n}\big(\sigma(\cdot),L\big)\right)\right\}d\mathcal{H}^{N-1}(x)<\delta\,,\tag{3.190}$$

where

$$E_{x,n}\big(\sigma(\cdot),L\big):=\int\limits_{I_N}\frac{1}{L}F\bigg(L^n\nabla^n\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,L^{n-1}\nabla^{n-1}\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,\ldots,\\L\nabla\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,\{\boldsymbol{A}\cdot\nabla\sigma\}\big(S_x(s)\big),\,\boldsymbol{B}\cdot v(x),\,\zeta\big(s_1,f^+(x),f^-(x)\big)\bigg)\,ds\,,\tag{3.191}$$

$$\zeta(s,a,b):=\begin{cases}a & \text{if } s>0\,,\\ b & \text{if } s<0\,,\end{cases}\tag{3.192}$$

$$\mathcal{W}_0^{(n)}\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big):=\\\Big\{u\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^d):\boldsymbol{A}\cdot\nabla u\in C^n(\mathbb{R}^N,\mathbb{R}^m),\ \ \boldsymbol{A}\cdot\nabla u(y)=(\boldsymbol{A}\cdot\nabla v)^-(x)\ \text{ if }\ y\cdot\boldsymbol{\nu}(x)\leq-1/2,\\\boldsymbol{A}\cdot\nabla u(y)=(\boldsymbol{A}\cdot\nabla v)^+(x)\ \text{ if }\ y\cdot\boldsymbol{\nu}(x)\geq1/2\ \text{ and }\ \boldsymbol{A}\cdot\nabla u\big(y+\boldsymbol{k}_j(x)\big)=\boldsymbol{A}\cdot\nabla u(y)\ \ \forall j=2,3,\ldots,N\Big\},\tag{3.193}$$

$$I_N:=\Big\{s\in\mathbb{R}^N:\ -1/2<s_j<1/2\quad\forall j=1,2,\ldots,N\Big\},\tag{3.194}$$

$\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\}$ *is an arbitrary system of linearly independent vectors in* $\mathbb{R}^N$ *satisfying* $\boldsymbol{k}_1(x)=\boldsymbol{\nu}(x)$ *and* $\boldsymbol{k}_j(x)\cdot\boldsymbol{\nu}(x)=0$ *for* $j\geq2$, *the linear transformation* $S_x(s)=S_x(s_1,s_2,\ldots,s_N):\mathbb{R}^N\to\mathbb{R}^N$ *is defined by*

$$S_x(s):=\sum_{j=1}^{N}s_j\,\boldsymbol{k}_j(x)\,,\tag{3.195}$$

and we assume that the orientation of J_f *coincides with the orientation of* $J_{\boldsymbol{A}\cdot\nabla v}$ $\mathcal{H}^{N-1}$ *a.e. on* $J_f\cap J_{\boldsymbol{A}\cdot\nabla v}$. *Moreover,* $\boldsymbol{A}\cdot\nabla v_\varepsilon$, $\varepsilon\nabla\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}$, $\varepsilon^2\nabla^2\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}$, $\ldots$, $\varepsilon^n\nabla^n\{\boldsymbol{A}\cdot\nabla v_\varepsilon\}$, $\boldsymbol{B}\cdot v_\varepsilon$ *and* $\nabla\{\boldsymbol{B}\cdot v_\varepsilon\}$ *are bounded in* L^∞ *sequences, there exists a compact* $K=K_\delta\subset\subset\Omega$ *such that* $v_\varepsilon(x)=\psi_\varepsilon(x)$ *for every* $0<\varepsilon<1$ *and every* $x\in\mathbb{R}^N\setminus K$, *where*

$$\psi_\varepsilon(x):=\frac{1}{\varepsilon^N}\Big\langle\eta\Big(\frac{y-x}{\varepsilon}\Big),v(y)\Big\rangle\,,\tag{3.196}$$

and

$$\limsup_{\varepsilon\to0^+}\left|\frac{1}{\varepsilon}\bigg(\int_\Omega\boldsymbol{A}\cdot\nabla v_\varepsilon(x)\,dx-\int_\Omega\{\boldsymbol{A}\cdot\nabla v\}(x)\,dx\bigg)\right|<+\infty\,,\tag{3.197}$$

4 The applications

Theorem 4.1. *Let* $\Omega\subset\mathbb{R}^N$ *be an open set. Furthermore, let*

$$F\in C^1\Big(\mathbb{R}^{k\times N\times N}\times\mathbb{R}^{d\times N\times N}\times\mathbb{R}^{m\times N}\times\mathbb{R}^{k\times N}\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\times\mathbb{R}^k\times\mathbb{R}^q\,,\,\mathbb{R}\Big)$$

satisfying $F\geq0$. *Let* $f\in BV_{loc}(\mathbb{R}^N,\mathbb{R}^q)\cap L^\infty$, $v\in Lip(\mathbb{R}^N,\mathbb{R}^k)\cap L^1\cap L^\infty$, $\bar{m}\in BV(\mathbb{R}^N,\mathbb{R}^{d\times N})\cap L^\infty$ *and* $\varphi\in BV(\mathbb{R}^N,\mathbb{R}^m)\cap L^\infty$ *be such that* $\nabla v\in BV(\mathbb{R}^N,\mathbb{R}^{k\times N})$, $\|D(\nabla v)\|(\partial\Omega)=0$, $\|D\bar{m}\|(\partial\Omega)=0$, $\|D\varphi\|(\partial\Omega)=0$, $div_x\bar{m}(x)=0$ *a.e. in* $\mathbb{R}^N$ *and* $F\big(0,0,0,\nabla v(x),\bar{m}(x),\varphi(x),v(x),f(x)\big)=0$ *a.e. in* Ω. *Then, for* $\eta\in\mathcal{V}_0$, *for*

every $\delta > 0$ there exist sequences $\{v_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N,\mathbb{R}^k)$, $\{m_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N,\mathbb{R}^{d\times N})$ and $\{\psi_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N,\mathbb{R}^m)$ such that $div_x m_\varepsilon(x) \equiv 0$ in $\mathbb{R}^N$, $\int_\Omega \psi_\varepsilon(x)\,dx = \int_\Omega \varphi(x)\,dx$, $\lim_{\varepsilon\to0^+} v_\varepsilon = v$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+}(v_\varepsilon - v)/\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^k)$, $\lim_{\varepsilon\to0^+} m_\varepsilon = \bar m$ in $L^p(\mathbb{R}^N,\mathbb{R}^{d\times N})$, $\lim_{\varepsilon\to0^+}\psi_\varepsilon = \varphi$ in $L^p(\mathbb{R}^N,\mathbb{R}^m)$, $\lim_{\varepsilon\to0^+}\varepsilon\nabla^2 v_\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{k\times N\times N})$, $\lim_{\varepsilon\to0^+}\varepsilon\nabla m_\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{d\times N\times N})$, $\lim_{\varepsilon\to0^+}\varepsilon\nabla\psi_\varepsilon = 0$ in $L^p(\mathbb{R}^N,\mathbb{R}^{m\times N})$ for every $p\geq 1$ and

$$0 \leq \lim_{\varepsilon\to0}\int_\Omega \frac{1}{\varepsilon}F\Big(\varepsilon\nabla^2 v_\varepsilon(x),\,\varepsilon\nabla m_\varepsilon(x),\,\varepsilon\nabla\psi_\varepsilon(x),\,\nabla v_\varepsilon(x),\,m_\varepsilon(x),\,\psi_\varepsilon(x),\,v_\varepsilon(x),\,f(x)\Big)\,dx \\ - \int_{\Omega\cap(J_{\nabla v}\cup J_{\bar m}\cup J_\varphi)}\Bigg(\inf\bigg\{\bar E_x\big(\sigma(\cdot),\theta(\cdot),\gamma(\cdot),L\big):\ \ L>0, \\ \sigma\in\mathcal{W}_0^{(1)}(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N),\,\theta\in\mathcal{W}_0^{(2)}(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N),\,\gamma\in\mathcal{W}_0^{(3)}(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)\bigg\}\Bigg)\,d\mathcal{H}^{N-1}(x) < \delta\,, \tag{4.1}$$

where

$$\bar E_x\big(\sigma(\cdot),\theta(\cdot),\gamma(\cdot),L\big) := \int_{I^+_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N}}\frac{1}{L}F\Big(L\nabla^2\sigma(y),\,L\nabla\theta(y),\,L\nabla\gamma(y),\,\nabla\sigma(y),\,\theta(y),\,\gamma(y),\,v(x),\,f^+(x)\Big)\,dy \\ + \int_{I^-_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N}}\frac{1}{L}F\Big(L\nabla^2\sigma(y),\,L\nabla\theta(y),\,L\nabla\gamma(y),\,\nabla\sigma(y),\,\theta(y),\,\gamma(y),\,v(x),\,f^-(x)\Big)\,dy\,, \tag{4.2}$$

$$\mathcal{W}_0^{(1)}\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big) := \bigg\{u\in C^2(\mathbb{R}^N,\mathbb{R}^k):\ \ \nabla u(y) = (\nabla v)^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq -1/2, \\ \nabla u(y) = (\nabla v)^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq 1/2\ \ \textit{and}\ \ \nabla u\big(y+\boldsymbol{k}_j(x)\big) = \nabla u(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.3}$$

$$\mathcal{W}_0^{(2)}\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big) := \\ \bigg\{\xi\in C^1(\mathbb{R}^N,\mathbb{R}^{d\times N}):\ \ div_y\xi(y)\equiv 0,\ \ \xi(y) = \bar m^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq -1/2, \\ \xi(y) = \bar m^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq 1/2\ \ \textit{and}\ \ \xi\big(y+\boldsymbol{k}_j(x)\big) = \xi(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.4}$$

$$\mathcal{W}_0^{(3)}\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big) := \bigg\{\zeta\in C^1(\mathbb{R}^N,\mathbb{R}^m):\ \ \zeta(y) = \varphi^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq -1/2, \\ \zeta(y) = \varphi^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq 1/2\ \ \textit{and}\ \ \zeta\big(y+\boldsymbol{k}_j(x)\big) = \zeta(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.5}$$

$$I^-_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N} := \Big\{y\in\mathbb{R}^N:\ -1/2 < y\cdot\boldsymbol{k}_1 < 0,\ \ |y\cdot\boldsymbol{k}_j| < 1/2\quad\forall j=2,3,\ldots,N\Big\}, \\ I^+_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N} := \Big\{y\in\mathbb{R}^N:\ 0 < y\cdot\boldsymbol{k}_1 < 1/2,\ \ |y\cdot\boldsymbol{k}_j| < 1/2\quad\forall j=2,3,\ldots,N\Big\}, \tag{4.6}$$

$\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\}$ is an orthonormal base in $\mathbb{R}^N$, satisfying $\boldsymbol{k}_1(x) = \boldsymbol{\nu}(x)$, and we assume that the orientations of $J_{\nabla v}$, $J_{\bar m}$, J_φ and J_f coincides $\mathcal{H}^{N-1}$ a.e. and given by the vector $\boldsymbol{\nu}(x)$. Moreover, ∇v_ε, $\varepsilon\nabla^2 v_\varepsilon$, v_ε, m_ε, $\varepsilon\nabla m_\varepsilon$, ψ_ε and $\varepsilon\nabla\psi_\varepsilon$ are bounded in L^∞ sequences, and there exists a compact $K = K_\delta \subset\subset \Omega$ such that $v_\varepsilon(x) = v_\varepsilon^{(0)}(x)$, $m_\varepsilon(x) = m_\varepsilon^{(0)}(x)$ and $\psi_\varepsilon(x) = \psi_\varepsilon^{(0)}(x)$ for every $0<\varepsilon<1$ and every $x\in\mathbb{R}^N\setminus K$, where

$$v_\varepsilon^{(0)}(x) = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,v(y)\,dy\,,\quad m_\varepsilon^{(0)}(x) = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,\bar m(y)\,dy\,, \\ \psi_\varepsilon^{(0)}(x) = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,\varphi(y)\,dy\,.$$

Proof. Define the Borel sets:

$$\mathcal{K}_j := \Big\{x \in \Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi) :\ \boldsymbol{\nu}(x) \cdot \boldsymbol{e}_j \neq 0\Big\} \qquad \forall j = 1, 2, \dots N\,, \tag{4.7}$$

$$\mathcal{A}_j := \left\{\boldsymbol{k} \in \mathbb{R}^N :\ \ \mathcal{H}^{N-1}\Big(\big\{x \in \mathcal{K}_j :\ \boldsymbol{\nu}(x) \cdot \boldsymbol{k} = 0\big\}\Big) > 0\right\} \qquad \forall j = 1, 2, \dots N\,, \tag{4.8}$$

$$\mathcal{A} := \left\{\boldsymbol{k} \in \mathbb{R}^N :\ \ \mathcal{H}^{N-1}\Big(\big\{x \in \Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi) :\ \boldsymbol{\nu}(x) \cdot \boldsymbol{k} = 0\big\}\Big) > 0\right\}, \tag{4.9}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N\}$ is the standard orthonormal base in $\mathbb{R}^N$. We will prove now that

$$\mathcal{L}^N(\mathcal{A}) = 0\,. \tag{4.10}$$

Indeed, since $\Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi) = \bigcup_{j=1}^N \mathcal{K}_j$, we have $\mathcal{A} = \bigcup_{j=1}^N \mathcal{A}_j$. Therefore it is sufficient to prove

$$\mathcal{L}^N(\mathcal{A}_j) = 0 \qquad \forall j = 1, 2, \dots N\,. \tag{4.11}$$

Without any loss of generality we can prove it only in the particular case $j = 1$. Then

$$\mathcal{A}_1 := \left\{(k_1, k') := \boldsymbol{k} \in \mathbb{R}^N :\ \ \mathcal{H}^{N-1}\left(\left\{x \in \mathcal{K}_1 :\ k_1 = -\frac{\nu'(x) \cdot k'}{\nu_1(x)}\right\}\right) > 0\right\},$$

where $\boldsymbol{k} = (k_1, k') \in \mathbb{R} \times \mathbb{R}^{N-1} = \mathbb{R}^N$ and $\boldsymbol{\nu}(x) = \big(\nu_1(x), \nu'(x)\big) \in \mathbb{R} \times \mathbb{R}^{N-1} = \mathbb{R}^N$. On the other hand set

$$\mathcal{B}(k') := \left\{k_1 \in \mathbb{R} :\ \ \mathcal{H}^{N-1}\left(\left\{x \in \mathcal{K}_1 :\ k_1 = -\frac{\nu'(x) \cdot k'}{\nu_1(x)}\right\}\right) > 0\right\} \qquad \forall k' \in \mathbb{R}^{N-1}\,.$$

Thus since

$$\left\{x \in \mathcal{K}_1 :\ a = -\frac{\nu'(x) \cdot k'}{\nu_1(x)}\right\} \cap \left\{x \in \mathcal{K}_1 :\ b = -\frac{\nu'(x) \cdot k'}{\nu_1(x)}\right\} = \emptyset \quad \text{if } a \neq b\,,$$

and since the set $\mathcal{K}_1$ is $\mathcal{H}^{N-1}$ σ-finite we obtain that for every $k' \in \mathbb{R}^{N-1}$ the set $\mathcal{B}(k')$ is at most countable. Therefore, for every $k' \in \mathbb{R}^{N-1}$ we have $\mathcal{L}^1\big(\mathcal{B}(k')\big) = 0$ and thus

$$\mathcal{L}^N(\mathcal{A}_1) = \int_{\mathbb{R}^{N-1}} \mathcal{L}^1\big(\mathcal{B}(k')\big) dk' = 0\,,$$

So we proved (4.11), which implies (4.10).

In particular, by (4.10) we deduce that $S^{N-1} \setminus \mathcal{A} \neq \emptyset$. So there exists $\boldsymbol{r}_0 \in S^{N-1} \setminus \mathcal{A}$ and we have

$$\mathcal{H}^{N-1}\Big(\big\{x \in \Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi) :\ \boldsymbol{\nu}(x) \cdot \boldsymbol{r}_0 = 0\big\}\Big) = 0\,. \tag{4.12}$$

Without loss of generality we may assume that $\boldsymbol{r}_0 = \boldsymbol{e}_1 := (1, 0, 0, \dots, 0)$. Therefore from this point we assume that

$$\mathcal{H}^{N-1}\Big(\big\{x \in \Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi) :\ \boldsymbol{\nu}(x) \cdot \boldsymbol{e}_1 = 0\big\}\Big) = 0\,, \tag{4.13}$$

i.e. $\nu_1(x) \neq 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar m} \cup J_\varphi)$. Next define $\Phi : \mathbb{R}^N \to \mathbb{R}^m$ and $M : \mathbb{R}^N \to \mathbb{R}^{d \times (N-1)}$ by

$$\Phi(x) := \int_{-\infty}^{x_1} \varphi(s, x') ds\,, \quad \text{and} \quad M(x) := \int_{-\infty}^{x_1} m'(s, x') ds \qquad \forall x = (x_1, x') := (x_1, x_2, \dots x_N) \in \mathbb{R}^N\,, \tag{4.14}$$

where we denote by $m_1(x) : \mathbb{R}^N \to \mathbb{R}^d$ and $m'(x) : \mathbb{R}^N \to \mathbb{R}^{d \times (N-1)}$ the first column and the rest of the matrix valued function $\bar m(x) : \mathbb{R}^N \to \mathbb{R}^{d \times N}$, so that $\big(m_1(x), m'(x)\big) := \bar m(x) : \mathbb{R}^N \to \mathbb{R}^{d \times N}$. Then, since $div_x \bar m \equiv 0$, by (4.14) we obtain

$$\frac{\partial \Phi}{\partial x_1}(x) = \varphi(x)\,, \quad \frac{\partial M}{\partial x_1}(x) = m'(x)\,, \quad \text{and} \quad -\, div_{x'} M(x) = m_1(x) \qquad \text{for a.e. } x = (x_1, x') \in \mathbb{R}^N\,. \tag{4.15}$$

Next for every $x \in \Omega \cap J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi$ define

$$Q_x(z) := \begin{cases} \varphi^+(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) \geq 0\,, \\ \varphi^-(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) < 0 \end{cases} \qquad P_x(z) := \begin{cases} \bar{m}^+(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) \geq 0\,, \\ \bar{m}^-(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) < 0 \end{cases}$$

$$\text{and} \quad H_x(z) := \begin{cases} (\nabla v)^+(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) \geq 0\,, \\ (\nabla v)^-(x) & \text{if} \quad z \cdot \boldsymbol{\nu}(x) < 0\,. \end{cases} \tag{4.16}$$

Since $div_x \bar{m} \equiv 0$ and $curl(\nabla v) \equiv 0$, clearly for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi)$ we have $div_z P(z) = 0$ and $curl_z H_x(z) = 0$. In particular, clearly exists a function $\gamma_x(z) : \mathbb{R}^N \to \mathbb{R}^k$ such that $\nabla_z \gamma_x(z) = H_x(z)$. Moreover, since $\nu_1(x) \neq 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi)$, then if $\nu_1(x) > 0$ we define

$$R_x(z) := \int_{-\infty}^{z_1} \Big(Q_x(s, z') - \varphi^-(x)\Big) ds \text{ and } D_x(z) := \int_{-\infty}^{z_1} \Big(P'_x(s, z') - \big\{m^-(x)\big\}'\Big) ds \quad \forall z = (z_1, z') \in \mathbb{R}^N\,, \tag{4.17}$$

where we denote by $Y_1 : \mathbb{R}^N \to \mathbb{R}^d$ and $Y' : \mathbb{R}^N \to \mathbb{R}^{d \times (N-1)}$ the first column and the rest of the matrix $Y : \mathbb{R}^N \to \mathbb{R}^{d \times N}$. Then for such x we will have

$$\begin{gathered} \frac{\partial}{\partial z_1}\Big(R_x(z) + z_1 \varphi^-(x)\Big) = Q(z), \quad \frac{\partial}{\partial z_1}\bigg(D_x(z) + z_1\{m'\}^-(x) - \frac{1}{N-1} m_1^-(x) \otimes z'\bigg) = P'_x(z), \\ - div_{z'}\bigg(D_x(z) + z_1\{m'\}^-(x) - \frac{1}{N-1} m_1^-(x) \otimes z'\bigg) = \{P_x\}_1(z), \quad \nabla_z \gamma_x(z) = H_x(z) \;\; \forall z = (z_1, z') \in \mathbb{R}^N. \end{gathered} \tag{4.18}$$

If $\nu_1(x) < 0$ we interchange the role of $(\varphi^-, \bar{m}^-)$ by $(\varphi^+, \bar{m}^+)$ in (4.17). Thus in general for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi)$ we have

$$\begin{gathered} \frac{\partial}{\partial z_1}\Big(R_x(z) + z_1 \varphi^\pm(x)\Big) = Q(z), \quad \frac{\partial}{\partial z_1}\bigg(D_x(z) + z_1\{m'\}^\pm(x) - \frac{1}{N-1} m_1^\pm(x) \otimes z'\bigg) = P'_x(z), \\ - div_{z'}\bigg(D_x(z) + z_1\{m'\}^\pm(x) - \frac{1}{N-1} m_1^\pm(x) \otimes z'\bigg) = \{P_x\}_1(z), \quad \nabla_z \gamma_x(z) = H_x(z) \;\; \forall z = (z_1, z') \in \mathbb{R}^N. \end{gathered} \tag{4.19}$$

Here we take the sign "$-$" if $\nu_1(x) > 0$ and the sign "$+$" if $\nu_1(x) < 0$.

Next for every $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi)$ fix an arbitrary system $\{\boldsymbol{k}_1(x), \boldsymbol{k}_2(x), \ldots \boldsymbol{k}_N(x)\}$ of linearly independent vectors in $\mathbb{R}^N$ satisfying $\boldsymbol{k}_1(x) = \boldsymbol{\nu}(x)$ and $\boldsymbol{k}_j(x) \cdot \boldsymbol{\nu}(x) = 0$ for $j \geq 2$. Then define

$$\mathcal{W}_1^{(1)}(x) := \bigg\{ u \in C^\infty(\mathbb{R}^N, \mathbb{R}^k) : \;\; \nabla u(y) = 0 \;\text{ if }\; |y \cdot \boldsymbol{\nu}(x)| \geq 1/2, \\ \text{and } \; \nabla u\big(y + \boldsymbol{k}_j(x)\big) = \nabla u(y) \;\; \forall j = 2, 3, \ldots, N \bigg\}, \tag{4.20}$$

$$\mathcal{W}_1^{(2)}(x) := \bigg\{ \xi \in C^\infty(\mathbb{R}^N, \mathbb{R}^{d \times N}) : \; div_y \xi(y) \equiv 0, \;\; \xi(y) = 0 \;\text{ if }\; |y \cdot \boldsymbol{\nu}(x)| \geq 1/2, \\ \text{and } \; \xi\big(y + \boldsymbol{k}_j(x)\big) = \xi(y) \;\; \forall j = 2, 3, \ldots, N \bigg\}, \tag{4.21}$$

$$\mathcal{W}_1^{(3)}(x) := \bigg\{ \zeta \in C^\infty(\mathbb{R}^N, \mathbb{R}^m) : \; \zeta(y) = 0 \;\text{ if }\; |y \cdot \boldsymbol{\nu}(x)| \geq 1/2, \\ \text{and } \; \zeta\big(y + \boldsymbol{k}_j(x)\big) = \zeta(y) \;\; \forall j = 2, 3, \ldots, N \bigg\}. \tag{4.22}$$

Then since $\nu_1(x) \neq 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap (J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi)$, for such fixed x and for every $\xi \in \mathcal{W}_1^{(2)}(x)$ and every $\zeta \in \mathcal{W}_1^{(3)}(x)$ we can define $\Xi_\zeta(y) : \mathbb{R}^N \to \mathbb{R}^m$ and $\Upsilon_\xi(y) : \mathbb{R}^N \to \mathbb{R}^{d\times(N-1)}$ by

$$\Xi_\zeta(y) := \int_{-\infty}^{y_1} \zeta(s, y')ds\,, \quad \text{and} \quad \Upsilon_\xi(y) := \int_{-\infty}^{x_1} \xi'(s, y')ds \qquad \forall y = (y_1, y') \in \mathbb{R}^N\,, \tag{4.23}$$

Then, since $div_y \zeta \equiv 0$, we obtain

$$\frac{\partial \Xi_\zeta}{\partial y_1}(y) = \zeta(y)\,, \quad \frac{\partial \Upsilon_\xi}{\partial y_1}(x) = \xi'(y)\,, \quad \text{and} \quad - div_{y'} \Upsilon_\xi(y) = \xi_1(y) \qquad \forall y = (y_1, y') \in \mathbb{R}^N\,. \tag{4.24}$$

Moreover clearly by the definition for all x such that $\nu_1(x) \neq 0$ i.e. for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap J_{\nabla v} \cup J_{\bar{m}} \cup J_\varphi$ we have $\Xi_\zeta \in C^\infty(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^m) \cap Lip(\mathbb{R}^N, \mathbb{R}^m)$, $\Upsilon_\xi \in C^\infty(\mathbb{R}^N, \mathbb{R}^{d\times(N-1)}) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^{d\times(N-1)}) \cap Lip(\mathbb{R}^N, \mathbb{R}^{d\times(N-1)})$ and $\Xi_\zeta\big(y + \boldsymbol{k}_j(x)\big) = \Xi_\zeta(y)$ $\forall j = 2, 3, \ldots, N$, $\Upsilon_\xi\big(y + \boldsymbol{k}_j(x)\big) = \Upsilon_\xi(y)$ $\forall j = 2, 3, \ldots, N$. On the other hand, for every $u(y) \in \mathcal{W}_1^{(1)}(x)$ we clearly have $u \in C^\infty(\mathbb{R}^N, \mathbb{R}^k) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^k) \cap Lip(\mathbb{R}^N, \mathbb{R}^k)$ and $u\big(y + \boldsymbol{k}_j(x)\big) = u(y)$ $\forall j = 2, 3, \ldots, N$. So

$$\begin{aligned} &\Xi_\zeta \in C^\infty(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty \cap Lip \quad \text{and} \quad \Xi_\zeta\big(y + \boldsymbol{k}_j(x)\big) = \Xi_\zeta(y) \;\; \forall j = 2, 3, \ldots, N\,, \\ &\Upsilon_\xi \in C^\infty(\mathbb{R}^N, \mathbb{R}^{d\times(N-1)}) \cap L^\infty \cap Lip \quad \text{and} \quad \Upsilon_\xi\big(y + \boldsymbol{k}_j(x)\big) = \Upsilon_\xi(y) \;\; \forall j = 2, \ldots, N\,, \\ &u \in C^\infty(\mathbb{R}^N, \mathbb{R}^k) \cap L^\infty \cap Lip \quad \text{and} \quad u\big(y + \boldsymbol{k}_j(x)\big) = u(y) \;\; \forall j = 2, 3, \ldots, N\,. \end{aligned} \tag{4.25}$$

Then using (4.19), (4.24), (4.25) and Theorem 3.2 we deduce that for every $\delta > 0$ there exist sequences $\{v_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^k)$, $\{M_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^{d\times(N-1)})$ and $\{\Psi_\varepsilon\}_{0<\varepsilon<1} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^m)$ such that if we denote

$$\frac{\partial \Psi_\varepsilon}{\partial x_1}(x) = \bar{\psi}_\varepsilon(x)\,, \quad \frac{\partial M_\varepsilon}{\partial x_1}(x) = m'_\varepsilon(x)\,, \quad \text{and} \quad - div_{x'} M_\varepsilon(x) = (m_\varepsilon)_1(x) \qquad \forall x = (x_1, x') \in \mathbb{R}^N\,. \tag{4.26}$$

then we will have

$$\begin{aligned} 0 \leq \lim_{\varepsilon\to 0} \int_\Omega \frac{1}{\varepsilon} F\Big(\varepsilon\nabla^2 v_\varepsilon(x),\, \varepsilon\nabla m_\varepsilon(x),\, \varepsilon\nabla\bar{\psi}_\varepsilon(x),\, \nabla v_\varepsilon(x),\, m_\varepsilon(x),\, \bar{\psi}_\varepsilon(x),\, v_\varepsilon(x),\, f(x)\Big)\, dx \\ - \int_{\Omega\cap(J_{\nabla v}\cup J_{\bar{m}}\cup J_\varphi)} \Bigg(\inf\bigg\{\bar{E}_x\big(\sigma(\cdot), \theta(\cdot), \gamma(\cdot), L\big) : \;\; L > 0, \\ \sigma \in \mathcal{W}_0^{(1)}(x, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N),\, \theta \in \mathcal{W}_0^{(2)}(x, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N),\, \gamma \in \mathcal{W}_0^{(3)}(x, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)\bigg\}\Bigg)\, d\mathcal{H}^{N-1}(x) < \delta\,, \end{aligned} \tag{4.27}$$

and $\lim_{\varepsilon\to 0^+} v_\varepsilon = v$ in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^k)$, $\lim_{\varepsilon\to 0^+}(v_\varepsilon - v)/\varepsilon = 0$ in L^p, $\lim_{\varepsilon\to 0^+} m_\varepsilon = \bar{m}$ in L^p, $\lim_{\varepsilon\to 0^+} \bar{\psi}_\varepsilon = \varphi$ in L^p, $\lim_{\varepsilon\to 0^+} \varepsilon\nabla^2 v_\varepsilon = 0$ in L^p, $\lim_{\varepsilon\to 0^+} \varepsilon\nabla m_\varepsilon = 0$ in L^p, $\lim_{\varepsilon\to 0^+} \varepsilon\nabla\bar{\psi}_\varepsilon = 0$ in L^p for every $p \geq 1$. Moreover, ∇v_ε, $\varepsilon\nabla^2 v_\varepsilon$, v_ε, m_ε, $\varepsilon\nabla m_\varepsilon$, $\bar{\psi}_\varepsilon$ and $\varepsilon\nabla\bar{\psi}_\varepsilon$ will be bounded in L^∞ sequences, for some compact $\bar{K} \subset\subset \Omega$ we will have $v_\varepsilon(x) = v_\varepsilon^{(0)}(x)$, $m_\varepsilon(x) = m_\varepsilon^{(0)}(x)$ and $\bar{\psi}_\varepsilon(x) = \psi_\varepsilon^{(0)}(x)$ for every $x \in \mathbb{R}^N \setminus \bar{K}$ and we will have

$$\limsup_{\varepsilon\to 0^+} \left| \frac{1}{\varepsilon}\bigg(\int_\Omega \bar{\psi}_\varepsilon(x)\, dx - \int_\Omega \varphi(x)\, dx \bigg) \right| < +\infty\,. \tag{4.28}$$

Finally we will slightly modify the sequence $\bar{\psi}_\varepsilon$, so that all the properties, presented above, will preserve and moreover the modified sequence ψ_ε will satisfy the additional constraint $\int_\Omega \psi_\varepsilon = \int_\Omega \varphi$. For this purpose we define

$$d_\varepsilon := \int_\Omega \varphi(x)\, dx - \int_\Omega \bar{\psi}_\varepsilon(x)\, dx\,. \tag{4.29}$$

Then by (4.28) there exists a constant $C_0 > 0$, independent on ε such that

$$|d_\varepsilon| \leq C_0 \varepsilon \quad \forall \varepsilon \in (0, 1)\,. \tag{4.30}$$

Now fix some smooth function $\lambda(x)\in C_c^\infty(\Omega,\mathbb{R})$, such that $\int_\Omega \lambda(x)dx=1$, and a compact set $K\subset\subset\Omega$ such that $\operatorname{supp}\lambda\subset K$ and $\bar K\subset K$. Then define $\psi_\varepsilon(x)\in C^\infty(\mathbb{R}^N,\mathbb{R}^m)$ by

$$\psi_\varepsilon(x):=\bar\psi_\varepsilon(x)+\lambda(x)d_\varepsilon \quad \forall x\in\mathbb{R}^N\ \forall\varepsilon\in(0,1)\,. \tag{4.31}$$

Thus clearly by (4.29) we have

$$\int_\Omega \psi_\varepsilon(x)\,dx=\int_\Omega\varphi(x)\,dx\,. \tag{4.32}$$

Moreover, clearly by the definition, $\lim_{\varepsilon\to0^+}\psi_\varepsilon=\varphi$ in L^p and $\lim_{\varepsilon\to0^+}\varepsilon\nabla\psi_\varepsilon=0$ in L^p for every $p\geq1$, ψ_ε and $\varepsilon\nabla\psi_\varepsilon$ are bounded in L^∞ sequences and $\psi_\varepsilon(x)=\psi_\varepsilon^{(0)}(x)$ for every $x\in\mathbb{R}^N\setminus K$. Thus for the final conclusion it is sufficient to prove

$$\lim_{\varepsilon\to0}\Bigg\{\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon+\varepsilon d_\varepsilon\otimes\nabla\lambda,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon+\lambda d_\varepsilon,\,v_\varepsilon,\,f\Big)\,dx-$$
$$\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon,\,v_\varepsilon,\,f\Big)\,dx\Bigg\}=0\,. \tag{4.33}$$

Indeed

$$\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon+\varepsilon d_\varepsilon\otimes\nabla\lambda,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon+\lambda d_\varepsilon,\,v_\varepsilon,\,f\Big)\,dx-$$
$$\int\limits_\Omega\frac{1}{\varepsilon}F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon,\,v_\varepsilon,\,f\Big)\,dx=$$
$$\int_0^1\int_\Omega\Big(d_\varepsilon\otimes\nabla\lambda\Big):D_1F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon+t\varepsilon d_\varepsilon\otimes\nabla\lambda,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon+t\lambda d_\varepsilon,\,v_\varepsilon,\,f\Big)dxdt$$
$$+\int_0^1\int_\Omega\frac{d_\varepsilon}{\varepsilon}\cdot\nabla_2F\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon+t\varepsilon d_\varepsilon\otimes\nabla\lambda,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon+t\lambda d_\varepsilon,\,v_\varepsilon,\,f\Big)\lambda\,dxdt\,. \tag{4.34}$$

Here D_1F is the gradient of F on the argument in the place of $\nabla\bar\psi_\varepsilon$ and ∇_2F is the gradient of F on the argument in the place of $\bar\psi_\varepsilon$. On the other hand $F(0,0,0,\nabla v,\bar m,\varphi,v,f)=0$ a.e. and therefore, since $F\geq0$ we also have $DF(0,0,0,\nabla v,\bar m,\varphi,v,f)=0$. Thus, while being uniformly bounded,

$$DF\Big(\varepsilon\nabla^2v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\bar\psi_\varepsilon+t\varepsilon d_\varepsilon\otimes\nabla\lambda,\,\nabla v_\varepsilon,\,m_\varepsilon,\,\bar\psi_\varepsilon+t\lambda d_\varepsilon,\,v_\varepsilon,\,f\Big)\to0\quad\text{a.e. in }\Omega\,.$$

Therefore, since λ has a compact support and since by (4.30) $d_\varepsilon/\varepsilon$ is bounded, we deduce that the r.h.s. of (4.34) goes to zero as $\varepsilon\to0^+$. So we proved (4.33). $\square$

By the same method, using Theorem 3.3, we can prove the following more general Theorem.

Theorem 4.2. *Let $\Omega\subset\mathbb{R}^N$ be an open set. Furthermore, let F be a C^1 function defined on*

$$\big\{\mathbb{R}^{k\times N^{n+1}}\times\mathbb{R}^{d\times N^{n+1}}\times\mathbb{R}^{m\times N^n}\big\}\times\ldots\times\big\{\mathbb{R}^{k\times N\times N}\times\mathbb{R}^{d\times N\times N}\times\mathbb{R}^{m\times N}\big\}\times\big\{\mathbb{R}^{k\times N}\times\mathbb{R}^{d\times N}\times\mathbb{R}^m\big\}\times\mathbb{R}^k\times\mathbb{R}^q,$$

taking values in $\mathbb{R}$ and satisfying $F\geq0$. Let $f\in BV_{loc}(\mathbb{R}^N,\mathbb{R}^q)\cap L^\infty$, $v\in Lip(\mathbb{R}^N,\mathbb{R}^k)\cap L^1\cap L^\infty$, $\bar m\in BV(\mathbb{R}^N,\mathbb{R}^{d\times N})\cap L^\infty$ and $\varphi\in BV(\mathbb{R}^N,\mathbb{R}^m)\cap L^\infty$ be such that $\nabla v\in BV(\mathbb{R}^N,\mathbb{R}^{k\times N})$, $\|D(\nabla v)\|(\partial\Omega)=0$, $\|D\bar m\|(\partial\Omega)=0$, $\|D\varphi\|(\partial\Omega)=0$, $div_x\bar m(x)=0$ a.e. in $\mathbb{R}^N$ and

$$F\Big(0,0,\ldots,0,\nabla v,\bar m,\varphi,v,f\Big)=0\quad\textit{a.e. in }\Omega\,.$$

Then, for $\eta\in\mathcal{V}_0$, for every $\delta>0$ there exist sequences $\{v_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^k)$, $\{m_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^{d\times N})$ and $\{\psi_\varepsilon\}_{0<\varepsilon<1}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^m)$ such that $div_xm_\varepsilon(x)\equiv0$ in $\mathbb{R}^N$, $\int_\Omega\psi_\varepsilon(x)\,dx=\int_\Omega\varphi(x)\,dx$, $\lim_{\varepsilon\to0^+}v_\varepsilon=v$ in

$W^{1,p}$, $\lim_{\varepsilon\to0^+}(v_\varepsilon - v)/\varepsilon = 0$ *in* L^p, $\lim_{\varepsilon\to0^+} m_\varepsilon = m$ *in* L^p, $\lim_{\varepsilon\to0^+}\psi_\varepsilon = \varphi$ *in* L^p, $\lim_{\varepsilon\to0^+}\varepsilon^j\nabla^{1+j}v_\varepsilon = 0$ *in* L^p, $\lim_{\varepsilon\to0^+}\varepsilon^j\nabla^j m_\varepsilon = 0$ *in* L^p, $\lim_{\varepsilon\to0^+}\varepsilon^j\nabla^j\psi_\varepsilon = 0$ *in* L^p *for every* $p\geq1$ *and any* $j\in\{1,\ldots,n\}$ *and*

$$0\leq \lim_{\varepsilon\to0}\int\limits_\Omega \frac{1}{\varepsilon}\times \\ F\Big(\big\{\varepsilon^n\nabla^{n+1}v_\varepsilon,\, \varepsilon^n\nabla^n m_\varepsilon,\, \varepsilon^n\nabla^n\psi_\varepsilon\big\},\,\ldots,\big\{\varepsilon\nabla^2 v_\varepsilon,\,\varepsilon\nabla m_\varepsilon,\,\varepsilon\nabla\psi_\varepsilon\big\},\,\big\{\nabla v_\varepsilon,\, m_\varepsilon,\,\psi_\varepsilon\big\},\, v_\varepsilon,\, f\Big)\,dx \\ -\int\limits_{\Omega\cap(J_{\nabla v}\cup J_{\bar m}\cup J_\varphi)}\Bigg(\inf\bigg\{\bar E^{(n)}_x\big(\sigma(\cdot),\theta(\cdot),\gamma(\cdot),L\big):\ \ L>0, \\ \sigma\in\mathcal{W}^{(1)}_n(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N),\,\theta\in\mathcal{W}^{(2)}_n(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N),\,\gamma\in\mathcal{W}^{(3)}_n(x,\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)\bigg\}\Bigg)\,d\mathcal{H}^{N-1}(x)<\delta\,, \tag{4.35}$$

where

$$\bar E^{(n)}_x\big(\sigma(\cdot),\theta(\cdot),\gamma(\cdot),L\big):= \\ \int\limits_{I^+_{\boldsymbol{k}_1,,\ldots,\boldsymbol{k}_N}}\frac{1}{L}F\bigg(\Big\{L^n\nabla^{n+1}\sigma(y),L^n\nabla^n\theta(y),L^n\nabla^n\gamma(y)\Big\},\ldots,\Big\{\nabla\sigma(y),\theta(y),\gamma(y)\Big\},v(x),f^+(x)\bigg)dy+ \\ \int\limits_{I^-_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N}}\frac{1}{L}F\bigg(\Big\{L^n\nabla^{n+1}\sigma(y),L^n\nabla^n\theta(y),L^n\nabla^n\gamma(y)\Big\},\ldots,\Big\{\nabla\sigma(y),\theta(y),\gamma(y)\Big\},v(x),f^-(x)\bigg)dy\,, \tag{4.36}$$

$$\mathcal{W}^{(1)}_n\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big):=\bigg\{u\in C^{n+1}(\mathbb{R}^N,\mathbb{R}^k):\ \ \nabla u(y)=(\nabla v)^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq-1/2, \\ \nabla u(y)=(\nabla v)^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq1/2\ \ \textit{and}\ \ \nabla u\big(y+\boldsymbol{k}_j(x)\big)=\nabla u(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.37}$$

$$\mathcal{W}^{(2)}_n\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big):= \\ \bigg\{\xi\in C^n(\mathbb{R}^N,\mathbb{R}^{d\times N}):\ \ div_y\xi(y)\equiv0,\ \ \xi(y)=\bar m^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq-1/2, \\ \xi(y)=\bar m^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq1/2\ \ \textit{and}\ \ \xi\big(y+\boldsymbol{k}_j(x)\big)=\xi(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.38}$$

$$\mathcal{W}^{(3)}_n\big(x,\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\big):=\bigg\{\zeta\in C^n(\mathbb{R}^N,\mathbb{R}^m):\ \ \zeta(y)=\varphi^-(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\leq-1/2, \\ \zeta(y)=\varphi^+(x)\ \ \textit{if}\ \ y\cdot\boldsymbol{\nu}(x)\geq1/2\ \ \textit{and}\ \ \zeta\big(y+\boldsymbol{k}_j(x)\big)=\zeta(y)\ \ \forall j=2,3,\ldots,N\bigg\}, \tag{4.39}$$

$$I^-_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N}:=\Big\{y\in\mathbb{R}^N:\ -1/2<y\cdot\boldsymbol{k}_1<0,\ \ |y\cdot\boldsymbol{k}_j|<1/2\quad\forall j=2,3,\ldots,N\Big\}, \\ I^+_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N}:=\Big\{y\in\mathbb{R}^N:\ 0<y\cdot\boldsymbol{k}_1<1/2,\ \ |y\cdot\boldsymbol{k}_j|<1/2\quad\forall j=2,3,\ldots,N\Big\}, \tag{4.40}$$

$\{\boldsymbol{k}_1(x),\boldsymbol{k}_2(x),\ldots\boldsymbol{k}_N(x)\}$ *is an orthonormal base in* $\mathbb{R}^N$ *satisfying* $\boldsymbol{k}_1(x)=\boldsymbol{\nu}(x)$ *and we assume that the orientations of* $J_{\nabla v}$, $J_{\bar m}$, J_φ *and* J_f *coincides* $\mathcal{H}^{N-1}$ *a.e. and given by the vector* $\boldsymbol{\nu}(x)$*. Moreover,* ∇v_ε, $\varepsilon^j\nabla^{j+1}v_\varepsilon$, v_ε, m_ε, $\varepsilon^j\nabla^j m_\varepsilon$, ψ_ε *and* $\varepsilon^j\nabla^j\psi_\varepsilon$ *are bounded in* L^∞ *sequences for every* $j\in\{1,\ldots,n\}$*, and there exists a compact* $K=K_\delta\subset\subset\Omega$ *such that* $v_\varepsilon(x)=v^{(0)}_\varepsilon(x)$*,* $m_\varepsilon(x)=m^{(0)}_\varepsilon(x)$ *and* $\psi_\varepsilon(x)=\psi^{(0)}_\varepsilon(x)$ *for every* $0<\varepsilon<1$ *and every* $x\in\mathbb{R}^N\setminus K$*, where*

$$v^{(0)}_\varepsilon(x)=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,v(y)\,dy\,,\quad m^{(0)}_\varepsilon(x)=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,\bar m(y)\,dy\,, \\ \psi^{(0)}_\varepsilon(x)=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\,\varphi(y)\,dy\,.$$

A Appendix: Notations and basic results about BV-functions

- For given a real topological linear space X we denote by X^* the dual space (the space of continuous linear functionals from X to $\mathbb{R}$).
- For given $h \in X$ and $x^* \in X^*$ we denote by $\langle h, x^* \rangle_{X \times X^*}$ the value in $\mathbb{R}$ of the functional x^* on the vector h.
- For given two normed linear spaces X and Y we denote by $\mathcal{L}(X;Y)$ the linear space of continuous (bounded) linear operators from X to Y.
- For given $A \in \mathcal{L}(X;Y)$ and $h \in X$ we denote by $A \cdot h$ the value in Y of the operator A on the vector h.
- For given two reflexive Banach spaces X, Y and $S \in \mathcal{L}(X;Y)$ we denote by $S^* \in \mathcal{L}(Y^*;X^*)$ the corresponding adjoint operator, which satisfies
$$\langle x, S^* \cdot y^* \rangle_{X \times X^*} := \langle S \cdot x, y^* \rangle_{Y \times Y^*} \qquad \text{for every } y^* \in Y^* \text{ and } x \in X\,.$$
- Given open set $G \subset \mathbb{R}^N$ we denote by $\mathcal{D}(G, \mathbb{R}^d)$ the real topological linear space of compactly supported $\mathbb{R}^d$-valued test functions i.e. $C_c^\infty(G, \mathbb{R}^d)$ with the usual topology.
- We denote $\mathcal{D}'(G, \mathbb{R}^d) := \{\mathcal{D}(G, \mathbb{R}^d)\}^*$ (the space of $\mathbb{R}^d$ valued distributions in G).
- Given $h \in \mathcal{D}'(G, \mathbb{R}^d)$ and $\delta \in \mathcal{D}(G, \mathbb{R}^d)$ we denote $< \delta, h >:= \langle \delta, h \rangle_{\mathcal{D}(G,\mathbb{R}^d) \times \mathcal{D}'(G,\mathbb{R}^d)}$ i.e. the value in $\mathbb{R}$ of the distribution h on the test function δ.
- Given a linear operator $\boldsymbol{A} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^k)$ and a distribution $h \in \mathcal{D}'(G, \mathbb{R}^d)$ we denote by $\boldsymbol{A} \cdot h$ the distribution in $\mathcal{D}'(G, \mathbb{R}^k)$ defined by
$$< \delta, \boldsymbol{A} \cdot h >:=< \boldsymbol{A}^* \cdot \delta, h > \qquad \forall \delta \in \mathcal{D}(G, \mathbb{R}^k).$$
- Given $h \in \mathcal{D}'(G, \mathbb{R}^d)$ and $\delta \in \mathcal{D}(G, \mathbb{R})$ by $< \delta, h >$ we denote the vector in $\mathbb{R}^d$ which satisfy $< \delta, h > \cdot \boldsymbol{e} :=< \delta \boldsymbol{e}, h >$ for every $\boldsymbol{e} \in \mathbb{R}^d$.
- For a $p \times q$ matrix A with ij-th entry a_{ij} we denote by $|A| = \left(\Sigma_{i=1}^p \Sigma_{j=1}^q a_{ij}^2\right)^{1/2}$ the Frobenius norm of A.
- For two matrices $A, B \in \mathbb{R}^{p \times q}$ with ij-th entries a_{ij} and b_{ij} respectively, we write $A : B := \sum_{i=1}^p \sum_{j=1}^q a_{ij} b_{ij}$.
- For a $p \times q$ matrix A with ij-th entry a_{ij} and for a $q \times d$ matrix B with ij-th entry b_{ij} we denote by $AB := A \cdot B$ their product, i.e. the $p \times d$ matrix, with ij-th entry $\sum_{k=1}^q a_{ik} b_{kj}$.
- We identify a $\boldsymbol{u} = (u_1, \ldots, u_q) \in \mathbb{R}^q$ with the $q \times 1$ matrix having $i1$-th entry u_i, so that for a $p \times q$ matrix A with ij-th entry a_{ij} and for $\boldsymbol{v} = (v_1, v_2, \ldots, v_q) \in \mathbb{R}^q$ we denote by $A\,\boldsymbol{v} := A \cdot \boldsymbol{v}$ the p-dimensional vector $\boldsymbol{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$, given by $u_i = \sum_{k=1}^q a_{ik} v_k$ for every $1 \le i \le p$.
- As usual A^T denotes the transpose of the matrix A.
- For $\boldsymbol{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$ and $\boldsymbol{v} = (v_1, \ldots, v_p) \in \mathbb{R}^p$ we denote by $\boldsymbol{u}\boldsymbol{v} := \boldsymbol{u} \cdot \boldsymbol{v} := \sum_{k=1}^p u_k v_k$ the standard scalar product. We also note that $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{u}$ as products of matrices.
- For $\boldsymbol{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$ and $\boldsymbol{v} = (v_1, \ldots, v_q) \in \mathbb{R}^q$ we denote by $\boldsymbol{u} \otimes \boldsymbol{v}$ the $p \times q$ matrix with ij-th entry $u_i v_j$ (i.e. $\boldsymbol{u} \otimes \boldsymbol{v} = \boldsymbol{u}\,\boldsymbol{v}^T$ as product of matrices).
- For any $p \times q$ matrix A with ij-th entry a_{ij} and $\boldsymbol{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$ we denote by $A \otimes \boldsymbol{v}$ the $p \times q \times d$ tensor with ijk-th entry $a_{ij} v_k$.

- Given a vector valued function $f(x) = \big(f_1(x), \dots, f_k(x)\big) : \Omega \to \mathbb{R}^k$ ($\Omega \subset \mathbb{R}^N$) we denote by Df or by $\nabla_x f$ the $k \times N$ matrix with ij-th entry $\frac{\partial f_i}{\partial x_j}$.
- Given a matrix valued function $F(x) := \{F_{ij}(x)\} : \mathbb{R}^N \to \mathbb{R}^{k\times N}$ ($\Omega \subset \mathbb{R}^N$) we denote by $div\, F$ the $\mathbb{R}^k$-valued vector field defined by $div\, F := (l_1, \dots, l_k)$ where $l_i = \sum\limits_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$.
- Given a matrix valued function $F(x) = \big\{f_{ij}(x)\big\}(1 \le i \le p,\, 1 \le j \le q) : \Omega \to \mathbb{R}^{p\times q}$ ($\Omega \subset \mathbb{R}^N$) we denote by DF or by $\nabla_x F$ the $p \times q \times N$ tensor with ijk-th entry $\frac{\partial f_{ij}}{\partial x_k}$.
- For every dimension d we denote by I the unit $d \times d$-matrix and by O the null $d \times d$-matrix.
- Given a vector valued measure $\mu = (\mu_1, \dots, \mu_k)$ (where for any $1 \le j \le k$, μ_j is a finite signed measure) we denote by $\|\mu\|(E)$ its total variation measure of the set E.
- For any μ-measurable function f, we define the product measure $f \cdot \mu$ by: $f \cdot \mu(E) = \int_E f\, d\mu$, for every μ-measurable set E.
- Throughout this paper we assume that $\Omega \subset \mathbb{R}^N$ is an open set.

In what follows we present some known results on BV-spaces. We rely mainly on the book [4] by Ambrosio, Fusco and Pallara. Other sources are the books by Hudjaev and Volpert [39], Giusti [20] and Evans and Gariepy [18]. We begin by introducing some notation. For every $\boldsymbol{\nu} \in S^{N-1}$ (the unit sphere in $\mathbb{R}^N$) and $R > 0$ we set

$$B_R^+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N \,:\, |y-x| < R,\, (y-x)\cdot\boldsymbol{\nu} > 0\}\,, \tag{A.1}$$
$$B_R^-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : |y-x| < R,\, (y-x)\cdot\boldsymbol{\nu} < 0\}\,, \tag{A.2}$$
$$H_+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} > 0\}\,, \tag{A.3}$$
$$H_-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} < 0\} \tag{A.4}$$

and

$$H^0_{\boldsymbol{\nu}} = \{y \in \mathbb{R}^N \,:\, y\cdot\boldsymbol{\nu} = 0\}\,. \tag{A.5}$$

Next we recall the definition of the space of functions with bounded variation. In what follows, $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Definition A.1. Let Ω be a domain in $\mathbb{R}^N$ and let $f \in L^1(\Omega, \mathbb{R}^m)$. We say that $f \in BV(\Omega, \mathbb{R}^m)$ if

$$\int_\Omega |Df| := \sup\bigg\{\int_\Omega \sum_{k=1}^m f_k \operatorname{div}\, \varphi_k\, d\mathcal{L}^N \,:\, \varphi_k \in C^1_c(\Omega, \mathbb{R}^N)\ \forall k,\ \sum_{k=1}^m |\varphi_k(x)|^2 \le 1\ \forall x \in \Omega\bigg\}$$

is finite. In this case we define the BV-norm of f by $\|f\|_{BV} := \|f\|_{L^1} + \int_\Omega |Df|$.

We recall below some basic notions in Geometric Measure Theory (see [4]).

Definition A.2. Let Ω be a domain in $\mathbb{R}^N$. Consider a function $f \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.
i) We say that x is a point of *approximate continuity* of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho(x)} |f(y)-z|\, dy}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0\,.$$

In this case z is called an *approximate limit* of f at x and we denote z by $\tilde{f}(x)$. The set of points of approximate continuity of f is denoted by G_f.
ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \ne b$ and

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho^+(x,\boldsymbol{\nu})} |f(y)-a|\, dy}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0, \quad \lim_{\rho\to 0^+} \frac{\int_{B_\rho^-(x,\boldsymbol{\nu})} |f(y)-b|\, dy}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0. \tag{A.6}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined by (A.6) up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f. At a point of approximate continuity x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

We recall the following results on BV-functions that we shall use in the sequel. They are all taken from [4]. In all of them Ω is a domain in $\mathbb{R}^N$ and f belongs to $BV(\Omega, \mathbb{R}^m)$.

Theorem A.1 (Theorems 3.69 and 3.78 from [4])**.**
i) $\mathcal{H}^{N-1}$-almost every point in $\Omega \setminus J_f$ is a point of approximate continuity of f.
ii) The set J_f is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set, oriented by $\boldsymbol{\nu}(x)$. In other words, J_f is σ-finite with respect to $\mathcal{H}^{N-1}$, there exist countably many C^1 hypersurfaces $\{S_k\}_{k=1}^{\infty}$ such that $\mathcal{H}^{N-1}\Big(J_f \setminus \bigcup\limits_{k=1}^{\infty} S_k\Big) = 0$, and for $\mathcal{H}^{N-1}$-almost every $x \in J_f \cap S_k$, the approximate jump vector $\boldsymbol{\nu}(x)$ is normal to S_k at the point x.
iii) $\big[(f^+ - f^-) \otimes \boldsymbol{\nu}_f\big](x) \in L^1(J_f, d\mathcal{H}^{N-1})$.

Theorem A.2 (Theorems 3.92 and 3.78 from [4])**.** *The distributional gradient Df can be decomposed as a sum of three Borel regular finite matrix-valued measures on Ω,*

$$Df = D^a f + D^c f + D^j f$$

with

$$D^a f = (\nabla f)\, \mathcal{L}^N \quad \text{and} \quad D^j f = (f^+ - f^-) \otimes \boldsymbol{\nu}_f \mathcal{H}^{N-1} \llcorner J_f\,.$$

D^a, D^c and D^j are called absolutely continuous part, Cantor and jump part of Df, respectively, and $\nabla f \in L^1(\Omega, \mathbb{R}^{m\times N})$ is the approximate differential of f. The three parts are mutually singular to each other. Moreover we have the following properties:
i) The support of $D^c f$ is concentrated on a set of $\mathcal{L}^N$-measure zero, but $(D^c f)(B) = 0$ for any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{N-1}$;
ii) $[D^a f]\big(f^{-1}(H)\big) = 0$ and $[D^c f]\big(\tilde{f}^{-1}(H)\big) = 0$ for every $H \subset \mathbb{R}^m$ satisfying $\mathcal{H}^1(H) = 0$.

Theorem A.3 (Volpert chain rule, Theorems 3.96 and 3.99 from [4])**.** *Let $\Phi \in C^1(\mathbb{R}^m, \mathbb{R}^q)$ be a Lipschitz function satisfying $\Phi(0) = 0$ if $|\Omega| = \infty$. Then, $v(x) = (\Phi \circ f)(x)$ belongs to $BV(\Omega, \mathbb{R}^q)$ and we have*

$$D^a v = \nabla\Phi(f)\, \nabla f\, \mathcal{L}^N,\ D^c v = \nabla\Phi(\tilde{f})\, D^c f,\ D^j v = \big[\Phi(f^+) - \Phi(f^-)\big] \otimes \boldsymbol{\nu}_f\, \mathcal{H}^{N-1} \llcorner J_f\,.$$

We also recall that the trace operator T is a continuous map from $BV(\Omega)$, endowed with the strong topology (or more generally, the topology induced by strict convergence), to $L^1(\partial\Omega, \mathcal{H}^{N-1} \llcorner \partial\Omega)$, provided that Ω has a bounded Lipschitz boundary (see [4, Theorems 3.87 and 3.88]).

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